

RICCI FLOW, KILLING SPINORS, AND T-DUALITY IN GENERALIZED GEOMETRY

MARIO GARCIA-FERNANDEZ

Dedicated to Nigel Hitchin on the occasion of his seventieth birthday.

ABSTRACT. We introduce a notion of Ricci flow in generalized geometry, extending a previous definition by Gualtieri on exact Courant algebroids. Special stationary points of the flow are given by solutions to first-order differential equations, the Killing spinor equations, which encompass special holonomy metrics with solutions of the Strominger system. Our main result investigates a method to produce new solutions of the Ricci flow and the Killing spinor equations. For this, we consider T-duality between possibly topologically distinct torus bundles endowed with Courant structures, and demonstrate that solutions of the equations are exchanged under this symmetry. As applications, we give a mathematical explanation of the *dilaton shift* in string theory and prove that the Strominger system is preserved by T-duality.

1. INTRODUCTION

In generalized geometry, as initiated by Hitchin [31], the condition of having zero torsion does not determine a metric connection uniquely. Consequently, curvature quantities are often difficult to tackle. In this work we study various curvature quantities in generalized geometry, along with natural partial differential equations for a generalized metric. Given a solution of the equations with a large isometry group, our prime motivation is an attempt to produce another solution in a manifold with possibly different topology.

We begin by introducing a notion of Ricci flow in an arbitrary Courant algebroid E , which extends Gualtieri's definition of the generalized Ricci flow on exact Courant algebroids, studied by Streets in [44]. One novelty of our definition is that the gauge-fixed versions of the flow are part of the geometric framework, via the family of divergence operators on the Courant algebroid (see Definition 2.3). When E admits a spinor bundle, the gauge-fixing condition can be recast more intrinsically in terms of a twisted version of Alekseev-Xu's Dirac generating operators [4].

The relationship between the Ricci flow and the Killing spinor equations in generalized geometry [23]—for which we give a general definition—is also studied, showing that the Killing spinors are special stationary points of the flow given by first-order differential equations (see Proposition 5.7). Solutions of the Killing spinor equations (5.6) are very interesting geometric objects, which encompass special holonomy metrics with solutions of the Strominger system of partial differential equations [33, 45].

We next investigate a method to produce new solutions of the Ricci flow and the Killing spinor equations based on T-duality, a relation between quantum field theories discovered by physicists. The idea has its origins in the literature on string theory, namely the work of Buscher [9, 10], which was developed further by Roček and Verlinde [39]. The method is to start with a given solution with abelian symmetries,

This work is funded by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 655162.

reduce the space, and then produce another solution with dual symmetries. Our main result relies on an important observation by Cavalcanti and Gualtieri [12], which states that T-duality can be viewed as an isomorphism between Courant structures on two possibly topologically distinct manifolds.

To state our main result, let us introduce some notation. Let E be a Courant algebroid over a principal T -bundle M , with T a torus, such that the T -action on M lifts to E preserving the Courant algebroid structure. We shall call (E, M, T) an equivariant Courant algebroid. The simple reduction of E by T , which we denote by

$$E/T \rightarrow B,$$

is a Courant algebroid over B , whose sheaf of sections is given by the invariant sections of E . Given now a pair of equivariant Courant algebroids (E, M, T) and $(\hat{E}, \hat{M}, \hat{T})$ over a common base

$$\begin{array}{ccc} M & & \hat{M} \\ & \searrow p & \swarrow \hat{p} \\ & B & \end{array}$$

we say they are dual if there exists an isomorphism of Courant algebroids between the simple reductions

$$\psi: E/T \rightarrow \hat{E}/\hat{T},$$

which we call the duality isomorphism. We state now in an informal way our main result, and refer to Theorem 6.5 for the details.

Theorem 1.1. *Invariant solutions of the Ricci flow and the Killing spinor equations are exchanged under the duality isomorphism.*

The definition of T-duality which we consider here is a straightforward generalization of the main implication of [12, Theorem 3.1], which states that topological T-duality for principal torus bundles—as defined by Bouwknegt, Evslin and Mathai [7]—induces a duality isomorphism. In addition to topological T-duality, it comprises as a particular case the heterotic T-duality for transitive Courant algebroids in [5]. We should mention that our proof works also when T and \hat{T} are substituted by arbitrary non-abelian groups, but we have not been able to find any interesting examples in this case. Based on our proof, we believe that Theorem 1.1 extends to a fairly general class of Poisson-Lie T-duals, in the sense of Klimčík and Ševera [36]. We thank Ševera for clarifications about the non-abelian setup.

One difficulty we have faced is that the curvature quantities which are primarily attached to a generalized metric $V_+ \subset E$ are not an invariant of V_+ , but rather depend on the choice of a metric-compatible torsion-free generalized connection D on E (which always exists, by Proposition 3.3). Thus, the first objective of this work is to define curvature quantities which only depend on the generalized metric V_+ upon a choice of divergence operator

$$\operatorname{div}: \Gamma(E) \rightarrow C^\infty(M).$$

A detailed analysis of this question leads us to the definition of the Ricci tensor for a pair $(V_+, \operatorname{div})$ which is independent of choices (see Definition 4.5). The relation with the Killing spinor equations is established in Proposition 5.7 via a spinorial formula for the Ricci tensor (4.5), that relies in an algebraic Bianchi identity (4.2) for the curvature of a torsion-free generalized connection. This is a remarkable property, as

the construction of such objects typically involves standard connections with skew-symmetric torsion in the tangent bundle of M .

As an application of our framework we revisit some aspects of topological T-duality for exact Courant algebroids. The novelty here is to understand the notion of duality for pairs (V_+, div) in this particular context (see Definition 6.2), with the upshot of a mathematical explanation of the *dilaton shift* in string theory, as originally observed by Buscher [9] (see Remark 5.5). It would be interesting to compare our general formula (6.4) for the dilaton shift with [16, Eq. (3.16)] in the context of non-abelian T-duality in physics.

The last part of this work is devoted to prove—in Theorem 7.5—that the solutions of the Strominger system of partial differential equations are preserved by heterotic T-duality [5]. These equations have its origins in supergravity in physics [33, 45], and they were first considered in the mathematics literature in a seminal paper by Li and Yau [37] (see [22] for a recent review). Our proof has two essential ingredients: firstly, the characterization of the Strominger system in terms of the Killing spinor equations in generalized geometry in [23] combined with Theorem 1.1, and, secondly, a general formula for the dilaton shift under T-duality in Proposition 6.8. Our result builds towards the definition of a Strominger-Yau-Zaslow version of mirror symmetry for the Strominger system, as proposed in [48].

It is interesting to notice that Proposition 5.8 combined with Theorem 6.5 implies that special holonomy metrics with a continuous abelian group of isometries are preserved by T-duality (cf. [32, 46]). Based on the details of our proof, we believe that this is also true for the non-abelian Poisson-Lie T-duality [36]. Despite the fact that compact manifolds with holonomy $SU(n)$, G_2 or $Spin(7)$ have no continuous symmetries, it would be worth exploring this perspective of the present work in the abundant local examples that exist in the literature (see e.g. [41]).

Inspiration for our results comes from previous work in the physics literature. Even though the theory we attempt to develop here is purely mathematical, this work got underway trying to understand some constructions in [15, 26]. We should mention that our result about the Strominger system in Theorem 7.5, although expected, is apparently new to physicists. In physical terms, we demonstrate that T-duality is a *solution generating technique* in heterotic supergravity.

Very recently, as we had just completed the proof of our main Theorem 6.5, an interesting paper by Ševera and Valach appeared [43], which contains similar results as our Proposition 4.4 and part *i*) of our Theorem 6.5. While the main focus is rather similar, the methods and the definition of the Ricci flow seem to be very different. We hope that our applications to the mathematical understading of the dilaton shift and to the Strominger system motivate our framework. The main result and ideas of this work were presented in September 2016, during the conference ‘Celebrating 30 years of Higgs bundles and 15 years of generalized geometry’ at the ICMAT in Madrid.

Acknowledgements. I am grateful to Nigel Hitchin for stimulating discussions and insight. The present work is motivated by his teaching during the lecture course ‘Topics in Generalized Geometry’, held at the Mathematical Institute in Oxford in 2012. I am indebted to Roberto Rubio, with whom I did some of the calculations in Proposition 2.10, and for his careful reading of the manuscript. I wish also to thank Luis Álvarez-Cónsul, Vicente Cortés, Marco Gualtieri, Pavol Ševera, Jeffrey Streets, and Daniel Waldram for helpful discussions and comments about the manuscript.

2. CONNECTIONS AND DIRAC GENERATING OPERATORS

2.1. Torsion and divergence of generalized connections. A *Courant algebroid* over a manifold M is a vector bundle $E \rightarrow M$ equipped with a fibrewise nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$, a bilinear bracket $[\cdot, \cdot]$ on the smooth sections $\Gamma(E)$, and a bundle map $\pi: E \rightarrow TM$ called the anchor, which satisfy the following conditions for all $e_1, e_2, e_3 \in \Gamma(E)$ and $f \in C^\infty(M)$

- (C1): $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]],$
- (C2): $\pi([e_1, e_2]) = [\pi(e_1), \pi(e_2)],$
- (C3): $[e_1, fe_2] = \pi(e_1)(f)e_2 + f[e_1, e_2],$
- (C4): $\pi(e_1)\langle e_2, e_3 \rangle = \langle [e_1, e_2], e_3 \rangle + \langle e_2, [e_1, e_3] \rangle,$
- (C5): $2[e_1, e_1] = \pi^*d\langle e_1, e_1 \rangle,$

where in the last equality we use $\langle \cdot, \cdot \rangle$ to identify E with E^* . We note that condition (C2) is redundant, as it follows from (C1) and (C3).

A *generalized connection* on the Courant algebroid E is an E -connection compatible with the inner product [4, 30], that is, a first-order differential operator

$$D: \Gamma(E) \rightarrow \Gamma(E^* \otimes E)$$

satisfying the Leibniz rule and compatible with $\langle \cdot, \cdot \rangle$:

$$\begin{aligned} D_{e_1}(fe_2) &= \pi(e_1)(f)e_2 + fD_{e_1}e_2, \\ \pi(e_1)\langle e_2, e_3 \rangle &= \langle D_{e_1}e_2, e_3 \rangle + \langle e_2, D_{e_1}e_3 \rangle. \end{aligned} \quad (2.1)$$

We will denote $D_{e_1}(e_2) = \iota_{e_1}D(e_2)$, where ι is the contraction operator of elements in E^* with elements in E .

The set of generalized connections on E , that we will denote by \mathcal{D} , has a structure of affine space modelled on the vector space

$$\Gamma(E^* \otimes \mathfrak{o}(E)),$$

where $\mathfrak{o}(E)$ denotes the bundle of skew-symmetric endomorphisms of E with respect to the bilinear form $\langle \cdot, \cdot \rangle$. To see this, given a (standard) orthogonal connection ∇^E on $(E, \langle \cdot, \cdot \rangle)$ we can construct a generalized connection on E

$$D_{e_1}e_2 = \nabla_{\pi(e_1)}^E e_2, \quad (2.2)$$

so that any other generalized connection on E is of the form $D + \chi$, for an element $\chi \in \Gamma(E^* \otimes \mathfrak{o}(E))$.

Definition 2.1. The torsion $T_D \in \Gamma(\Lambda^3 E^*)$ of a generalized connection D on E is defined by

$$T_D(e_1, e_2, e_3) = \langle D_{e_1}e_2 - D_{e_2}e_1 - [e_1, e_2], e_3 \rangle + \langle D_{e_3}e_1, e_2 \rangle. \quad (2.3)$$

Using the skew-symmetrized (Courant) bracket $[[e_1, e_2]] = \frac{1}{2}([e_1, e_2] - [e_2, e_1])$ and the compatibility of D with the bilinear form on E , it is immediate to see that this definition is equivalent to Gualtieri's torsion, as defined in [30]. A less direct (although elementary) calculation shows that Definition 2.1 is equivalent to Alexseev-Xu's [4], given by the formula

$$T_D(e_1, e_2, e_3) = c.p. \left(\frac{1}{2} \langle D_{e_1}e_2 - D_{e_2}e_1, e_3 \rangle - \frac{1}{3} \langle [[e_1, e_2]], e_3 \rangle \right), \quad (2.4)$$

where *c.p.* stands for sum over cyclic permutations. We note that formula (2.4) is more general than (2.3), as it defines an element in $\Gamma(\Lambda^3 E^*)$ also for E -connections which are not compatible with the ambient metric.

Let T be an element in $\Gamma(\Lambda^3 E^*)$, and consider the set \mathcal{D}^T of generalized connections on E with fixed torsion T

$$\mathcal{D}^T \subset \mathcal{D}.$$

For a choice of $D \in \mathcal{D}^T$ and $\chi \in \Gamma(E^* \otimes \mathfrak{o}(E))$, the condition for $D' = D + \chi$ to be in \mathcal{D}^T is given by

$$c.p. \langle \chi_{e_1} e_2, e_3 \rangle = 0, \quad (2.5)$$

where the left hand side corresponds to the total skew-symmetrization of $\langle \chi_{e_1} e_2, e_3 \rangle$. Using the canonical identification $E \otimes \Lambda^2 E \cong E^* \otimes \mathfrak{o}(E)$, we can regard the affine space \mathcal{D}^T as modelled on the following vector space of mixed symmetric 3-tensors

$$\Sigma = \{\sigma \in \Gamma(E^{\otimes 3}) : \sigma_{123} = -\sigma_{132}, \text{ c.p. } \sigma_{123} = 0\}, \quad (2.6)$$

where $\sigma_{123} = \sigma(e_1, e_2, e_3)$ for $e_i \in \Gamma(E)$ (using $\langle \cdot, \cdot \rangle$ to identify $E \cong E^*$). Equivalently, considering the canonical exact sequence

$$0 \longrightarrow S^3 E \longrightarrow S^2 E \otimes E \longrightarrow E \otimes \Lambda^2 E \longrightarrow \Lambda^3 E \longrightarrow 0, \quad (2.7)$$

where $S^k E$ denotes the k -th symmetric product, we have

$$\Sigma \cong \Gamma((S^2 E \otimes E)/S^3 E).$$

From the previous sequence we observe that there is a distinguished subspace

$$E \subset S^2 E \otimes E : e \rightarrow \langle \cdot, \cdot \rangle^{-1} \otimes e$$

inducing a direct sum decomposition

$$S^2 E \otimes E = (S^2 E \otimes E)_0 \oplus E,$$

where $(S^2 E \otimes E)_0$ is the subbundle generated by elements of the form

$$(S^2 E \otimes E)_0 = \langle (e_1 \odot e_2) \otimes e : (e_1, e_2) = 0 \rangle.$$

From this, we obtain a splitting

$$\Sigma = \Sigma_0 \oplus \Gamma(E) \quad (2.8)$$

where $e \in \Gamma(E)$ corresponds to the mixed symmetric tensor σ^e defined by

$$\sigma^e(e_1, e_2, e_3) = \langle e_1, e_2 \rangle \langle e, e_3 \rangle - \langle e, e_2 \rangle \langle e_1, e_3 \rangle$$

and the complement of $\Gamma(E)$ is given by

$$\Sigma_0 = \left\{ \sigma \in \Sigma : \sum_{i=1}^{r_E} \sigma(e_i, \tilde{e}_i, \cdot) = 0 \right\}. \quad (2.9)$$

Here, r_E denotes the rank of E , $\{e_i\}$ is an orthogonal local frame for E and the $\{\tilde{e}_i\}$ are sections of E defined so that $\langle e_i, \tilde{e}_j \rangle = \delta_{ij}$. More generally, given an arbitrary element $\chi \in \Gamma(E^* \otimes \mathfrak{o}(E))$ we have a unique decomposition

$$\chi = \chi_0 + \chi^e, \quad (2.10)$$

where $\chi_0 \in \Gamma(E^* \otimes \mathfrak{o}(E))$ is such that

$$\sum_{i=1}^{r_E} (\chi_0)_{e_i} \tilde{e}_i = 0,$$

and

$$\chi_{e_1}^e e_2 = \langle e_1, e_2 \rangle e - \langle e, e_2 \rangle e_1$$

with

$$e = \frac{1}{r_E - 1} \sum_{i=1}^{r_E} \chi_{e_i} \tilde{e}_i.$$

As observed in [21], the E^* -valued skew-symmetric endomorphism χ^e in the decomposition (2.10) is reminiscent of the ‘1-form valued Weyl endomorphisms’ in pseudo-Riemannian geometry, which appear in the variation of a metric connection with fixed torsion upon a conformal change of the metric. Similarly, our (E^* -valued) *Weyl endomorphisms* χ^e enable us to deform a generalized connection D with fixed torsion T inside the space \mathcal{D}^T . Notice that the notion of generalized connection—considered here—does not allow for conformal changes of the ambient metric $\langle \cdot, \cdot \rangle$.

For later applications in this work, it is important to ‘gauge-fix’ the Weyl degrees of freedom in the space of generalized connections \mathcal{D}^T corresponding to $\Gamma(E)$ in (2.8). We will call this procedure *Weyl fixing*. First, we recall the notion of *divergence* of a generalized connection introduced in [4]. Given a generalized connection D on E and $e \in \Gamma(E)$, we can define an endomorphism De of E by $e' \rightarrow D_{e'}e$.

Definition 2.2 ([4]). The *divergence* of $e \in \Gamma(E)$ with respect to a generalized connection D is

$$\operatorname{div}_D(e) = \operatorname{tr} De \in C^\infty(M). \quad (2.11)$$

The divergence of a generalized connection defines a first-order differential operator

$$\operatorname{div}_D: \Gamma(E) \rightarrow C^\infty(M)$$

satisfying the π -Leibniz rule

$$\operatorname{div}_D(fe) = \pi(e)(f) + f \operatorname{div}_D e, \quad (2.12)$$

for any $f \in C^\infty(M)$. For our applications, it will be useful to have a definition independent of the choice of a generalized connection, which we shall give now.

Definition 2.3. A differential operator $\operatorname{div}: \Gamma(E) \rightarrow C^\infty(M)$ satisfying (2.12) will be called a *divergence operator* on E .

Note that the divergence operators on E form an affine space modelled on $\Gamma(E^*)$. Our next result shows that the divergence constrains the Weyl degrees of freedom in the space generalized connections with fixed torsion, as required.

Lemma 2.4. Let $T \in \Gamma(\Lambda^3 E^*)$ and $\operatorname{div}: \Gamma(E) \rightarrow C^\infty(M)$ be a divergence operator on E . Then

$$\mathcal{D}^T(\operatorname{div}) = \{D \in \mathcal{D} \mid T_D = T, \operatorname{div}_D = \operatorname{div}\} \subset \mathcal{D}^T$$

is an affine space modelled on Σ_0 , as defined in (2.9).

Proof. Given $D', D \in \mathcal{D}^T$, we denote $D' = D + \chi$ with $\chi \in \Sigma$ (see (2.6)). Then, by definition of the divergence of $e' \in \Gamma(E)$ we have

$$\operatorname{div}_{D'}(e') = \operatorname{div}_D(e) - \sum_{i=1}^{r_E} \langle \chi_{\tilde{e}_i} e_i, e' \rangle. \quad (2.13)$$

Decomposing $\chi = \chi_0 + \chi^e$ as in (2.10), for $e \in \Gamma(E)$, we obtain

$$\sum_{i=1}^{r_E} \chi_{\tilde{e}_i} e_i = (r_E - 1)e.$$

Imposing now $\operatorname{div}_{D'} = \operatorname{div}_D = \operatorname{div}$, implies $e = 0$, which concludes the proof. \square

Example 2.5. Let $E = T \oplus T^*$ be an exact Courant algebroid over M with pairing

$$\langle X + \xi, X + \xi \rangle = \xi(X),$$

anchor $\pi(X + \xi) = X$, and bracket

$$[X + \xi, Y + \eta] = [X, Y] + L_X \eta - \iota_Y d\xi + \iota_Y \iota_X H,$$

for $H \in \Gamma(\Lambda^3 T^*)$ a closed 3-form. Given a connection ∇ on T , we consider the generalized connection D on E induced by the orthogonal connection $\nabla \oplus \nabla^*$ as in (2.2), where ∇^* denotes the induced connection on the cotangent bundle T^* . Then

$$T_D(e_1, e_2, e_3) = -\frac{1}{2} \pi^* H(e_1, e_2, e_3) + c.p. \langle s(\pi^* T_\nabla(e_1, e_2)), e_3 \rangle, \quad (2.14)$$

where $s: T \rightarrow E$ denotes the isotropic splitting $s(X) = X$ and T_∇ is the torsion of the connection ∇ , and

$$\operatorname{div}_D(e) = \operatorname{tr} \nabla X,$$

for $e = X + \xi$. Assuming now $T_\nabla = 0$, so that $\operatorname{tr} \nabla X = \operatorname{tr} \nabla_X - L_X$, and that there exists a density $\mu \in \Gamma(|\det T^*|)$ preserved by ∇ , we obtain

$$\operatorname{div}_D(e)\mu = L_X \mu,$$

which recovers the standard notion of divergence for the vector field $\pi(e) = X$.

2.2. Dirac generating operators and Weyl fixing. The aim of this section is to provide a different point of view on the Weyl fixing condition in Lemma 2.4. In order to do this, we build on the spinorial geometry of E and the theory of *Dirac generating operators* in [4, 27]. We hope that our discussion motivates the use of torsion-free generalized connections in the rest of this work.

Let $Cl(E)$ denote the bundle of Clifford algebras of the orthogonal bundle $(E, \langle \cdot, \cdot \rangle)$, defined by the relation

$$e^2 = \langle e, e \rangle$$

for any section $e \in \Gamma(E)$. In the rest of this section, we assume that $Cl(E)$ admits an irreducible Clifford module S , so that the Clifford action of sections of $Cl(E)$ on sections of S induces an isomorphism $\Gamma(\operatorname{End} S) \cong \Gamma(Cl(E))$. We will assume that the *spinor bundle* S is real.

The \mathbb{Z}_2 -grading on $\Gamma(Cl(E))$ induces a \mathbb{Z}_2 -grading on the differential operators on $\Gamma(S)$, such that the functions $f \in C^\infty(M)$ acting by multiplication on $\Gamma(S)$ are even operators. Consider the graded commutator on the space of differential operators on $\Gamma(S)$, given by

$$[L_0, L_1] = L_1 L_2 - (-1)^{\deg L_1 \deg L_2} L_2 L_1.$$

Definition 2.6 ([4]). A first-order odd differential operator \not{d} on $\Gamma(S)$ is called a *Dirac generating operator* of the Courant algebroid E if it satisfies the following conditions for every $f \in C^\infty(M)$ and $e_1, e_2 \in \Gamma(E)$

- i) $[[\not{d}, f], e \cdot] = \pi(e)(f),$
- ii) $[[\not{d}, e_1 \cdot], e_2 \cdot] = [e_1, e_2] \cdot,$
- iii) $\not{d}^2 \in C^\infty(M).$

According to the previous definition, a Dirac generating operator indeed generates the Courant algebroid structure—that is, the bracket and the anchor map—from the orthogonal bundle $(E, \langle \cdot, \cdot \rangle)$. The existence of a Dirac generating operator (or simply a generating operator) for a given Courant algebroid is nontrivial and was one of the main results of [4]. Furthermore, in [27] it was proved that, provided that it exists, there is a canonical choice compatible with *Weyl quantization*. In the following theorem, we summarize the result of this theory that we will need. Let us denote by $|\det T^*|^{1/2}$ the trivializable line bundle of half-densities on M . We denote by r_S the rank of S .

Theorem 2.7 ([4, 27]). *Assume that the line bundle $(\det S^*)^{\frac{1}{rs}}$ exists, and consider the twisted spinor bundle*

$$\mathbb{S} = S \otimes (\det S^*)^{\frac{1}{rs}} \otimes |\det T^*|^{\frac{1}{2}}. \quad (2.15)$$

Then, there exists a canonical generating operator \not{D}_0 on $\Gamma(\mathbb{S})$ of E . Furthermore, any other generating operator \not{D} is of the form $\not{D} = \not{D}_0 + e \cdot$, for $e \in \Gamma(E)$ such that $[\not{D}_0, e \cdot] \in C^\infty(M)$.

The main idea to construct a generating operator on $\Gamma(\mathbb{S})$ is to start with a Dirac operator \not{D} on $\Gamma(S)$ for a choice of generalized connection D , and then try to render the construction independent of choices. The definition of \not{D} involves a choice of spin E -connection, and this ambiguity is fixed by the twist $(\det S^*)^{\frac{1}{rs}}$. Another ambiguity comes from the torsion of D , regarded as an operator on $\Gamma(S)$ by Clifford multiplication. Finally, once the torsion is fixed, there is still an ambiguity coming from the Weyl endomorphisms in decomposition (2.8), which is removed by the twist $|\det T^*|^{\frac{1}{2}}$ and the divergence operator.

In this context, the *Weyl fixing* in Lemma 2.4 amounts to choosing a generating operator \not{D} in the space of sections of a different twisted spinor bundle, namely

$$\mathcal{S} = S \otimes (\det S^*)^{\frac{1}{rs}}, \quad (2.16)$$

and to constructing a space of generalized connections D on E with fixed Dirac operator $\not{D} = \not{D}_0$. An interesting consequence of this condition is that D is forced to be torsion-free, which motivates the use of these type of generalized connections in the present work. Note here that, since $|\det T^*|^{\frac{1}{2}}$ is trivializable, a generating operator on $\Gamma(\mathcal{S})$ always exists by Theorem 2.7. Before going into any details of our construction, let us discuss briefly a basic example.

Example 2.8. Let $E = T \oplus T^*$ be an exact Courant algebroid over M , as considered in Example 2.5. For a choice of Lagrangian $T^* \subset E$, we have a spinor bundle $S = \Lambda^* T^*$, with Clifford action

$$(X + \xi) \cdot \alpha = \iota_X \alpha + \xi \wedge \alpha, \quad (2.17)$$

where $\alpha \in \Gamma(\Lambda^* T^*)$. Then, $\det S^* \cong |\det T|^{\frac{rs}{2}}$ and the twisted spinor bundles are $\mathcal{S} = \Lambda^* T^* \otimes |\det T|^{\frac{1}{2}}$ and

$$\mathbb{S} = \Lambda^* T^* \otimes (\det T \otimes \det T^*)^{\frac{1}{2}} \cong \Lambda^* T^*,$$

with canonical generating operator on \mathbb{S} given by

$$\not{D}_0 = d - H \wedge.$$

If \not{D} is another generating operator on \mathbb{S} , then $\not{D} = \not{D}_0 + e \cdot$ and $[\not{D}_0, e \cdot] = 0$. Setting $e = X + \xi$, we have

$$[\not{D}_0, e \cdot] = L_X + (\iota_X H + d\xi) \wedge,$$

and therefore $\pi(e) = X = 0$ and $d\xi = 0$. Choosing now a trivialization $\mu_0^{\frac{1}{2}}$ of $|\det T^*|^{\frac{1}{2}}$, any other trivialization is of the form $\mu^{\frac{1}{2}} = e^f \mu_0^{\frac{1}{2}} \in \Gamma(|\det T^*|^{\frac{1}{2}})$ for a smooth function $f \in C^\infty(M)$, and therefore using the canonical generating operator on \mathbb{S} we have a generating operator \not{D} on \mathcal{S} corresponding to $\mu^{\frac{1}{2}}$:

$$\not{D} = e^{-f} \not{D}_0 e^f = \not{D}_0 = d - (H - df) \wedge.$$

In conclusion, the generating operators on $\Gamma(\mathbb{S})$ are parameterized by closed 1-forms, while the generating operators on $\Gamma(\mathcal{S})$ given by pull-back of \not{d}_0 form an affine space modelled on exact 1-forms.

Given a generalized connection D on E , we have an induced E -connection on $Cl(E)$, which we denote also by

$$D: \Gamma(Cl(E)) \rightarrow \Gamma(E^* \otimes Cl(E)).$$

An E -connection on S compatible with D is a first differential operator

$$D^S: \Gamma(S) \rightarrow \Gamma(E^* \otimes S)$$

satisfying the Leibniz rule with respect to the anchor map π (cf. (2.1)) and such that

$$D^S(e \cdot \alpha) = D(e) \cdot \alpha + e \cdot D^S \alpha.$$

Given an E -connection on S , there is an induced Dirac operator

$$\not{D}^S: \Gamma(S) \rightarrow \Gamma(S),$$

defined explicitly by

$$\not{D}^S \alpha = \frac{1}{2} \sum_{i=1}^{r_E} \tilde{e}_i \cdot D_{e_i}^S \alpha$$

for any orthogonal local frame $\{e_i\}$ of E , where \cdot denotes Clifford multiplication. A pair of E -connections on S compatible with the same generalized connection D differ by an element $e \in \Gamma(E)$, regarded as the E^* -valued endomorphisms of S given by $\alpha \rightarrow \langle e, \cdot \rangle \otimes \alpha$ (see [27, Lemma 3.4]). From this, it follows that D induces canonically a Dirac operator on the twisted spinor bundle \mathcal{S} (see (2.16))

$$\not{D}: \Gamma(\mathcal{S}) \rightarrow \Gamma(\mathcal{S}).$$

Definition 2.9. Let $\not{d}: \Gamma(\mathcal{S}) \rightarrow \Gamma(\mathcal{S})$ be a generating operator of the Courant algebroid E . The family of generalized connections associated to \not{d} is defined by:

$$\mathcal{D}(\not{d}) = \{D \in \mathcal{D} \mid \not{D} = \not{d}\}. \quad (2.18)$$

In the next lemma we prove that $\mathcal{D}(\not{d})$ provides a particular example of the family of Weyl-fixed generalized connections in Lemma 2.4.

Proposition 2.10. *The space $\mathcal{D}(\not{d})$ is non-empty, and all its elements are torsion-free. Furthermore, with the notation in Lemma 2.4, there is an equality $\mathcal{D}(\not{d}) = \mathcal{D}^0(\text{div}_D)$ for any choice of $D \in \mathcal{D}(\not{d})$.*

Proof. The proof follows closely the construction of generating Dirac operators in [4]. Let D be an arbitrary generalized connection on E . Note that the divergence defines an E -connection on $\det T^*$ by

$$D_{e_1}^\Lambda \mu = L_{\pi(e_1)} \mu - \text{div}_D(e) \mu, \quad (2.19)$$

where $\mu \in \Gamma(\det T^*)$. Choose a trivialization $\mu^{\frac{1}{2}}$ of $|\det T^*|^{\frac{1}{2}}$, inducing an isomorphism $\mathcal{S} \cong \mathbb{S}$, and define $e \in \Gamma(E)$ by

$$\langle e, \cdot \rangle = -\frac{1}{2} \mu^{-\frac{1}{2}} D^\Lambda \mu^{\frac{1}{2}} \in \Gamma(E).$$

Then, by [27, Proposition 5.12], the following formula defines the canonical generating operator on $\Gamma(\mathcal{S}) \cong \Gamma(\mathbb{S})$

$$\not{d}_0 = \not{D} + \frac{1}{4} T_D \cdot + e \cdot. \quad (2.20)$$

Let $\not{d} = \not{d}_0 + e' \cdot$ be an arbitrary generating operator on $\Gamma(\mathcal{S})$. Using the metric $\langle \cdot, \cdot \rangle$, we regard the torsion T_D as an element in $\Gamma(E^* \otimes \mathfrak{o}(E))$ and consider $D' = D - \frac{1}{3}T_D$, so that $T_{D'} = 0$ (see (2.5)) and $\not{d} = \not{D}' + e'' \cdot$, where $e'' = e + e'$. Finally, setting $D'' = D' + \chi^{e''}$ we have

$$\not{D}'' = \not{D}' + \frac{r_E - 1}{4}e'' \cdot. \quad (2.21)$$

and hence for $e'' = -4e''/(r_E - 1)$ we obtain $\not{d} = \not{D}''$.

Assume now that $\not{d} = \not{D}$ for a generating operator \not{d} on $\Gamma(\mathcal{S})$ and a generalized connection D on E . Using *Rubio's formula* for the torsion [40], given by

$$\iota_{e_2}\iota_{e_1}T_D \cdot = [[\not{D}, e_1 \cdot], e_2 \cdot] - [e_1, e_2] \cdot$$

for $e_1, e_2 \in \Gamma(E)$, it follows from property *ii*) in Definition 2.6 that $T_D = 0$. Given $D' = D + \chi$ with $\chi \in \Gamma(E^* \otimes \mathfrak{o}(E))$, we denote

$$\sigma(e_1, e_2, e_3) = \langle \chi_{e_1}e_2, e_3 \rangle$$

and let $c.p.(\sigma)$ be its total skew-symmetrization. Then, a direct calculation shows that

$$\not{D}' = \not{D} - \frac{1}{4}c.p.(\sigma) \cdot - \frac{1}{4} \sum_{i=1}^{r_E} \chi_{\tilde{e}_i}e_i \cdot. \quad (2.22)$$

for $e' \in \Gamma(E)$. Decomposing $\chi = \chi_0 + \chi^e$ as in (2.10), for $e \in \Gamma(E)$, yields

$$\sum_{i=1}^{r_E} \chi_{\tilde{e}_i}e_i = (r_E - 1)e.$$

Imposing now $\not{D}' = \not{D} = \not{d}$, we obtain $c.p.(\sigma) = 0$ and therefore $e = 0$ by (2.22). Finally, (2.13) implies that $\text{div}_{D'}(e') = \text{div}_D(e')$, which concludes the proof. \square

Example 2.11. Consider the exact Courant algebroid E endowed with the generalized connection D in Example 2.5. Assume that ∇ is torsion-free and that it preserves the density $\mu \in \Gamma(|\det T^*|)$. Then, we have

$$D_e^\Lambda \mu = \nabla_{\pi(e)} \mu,$$

and an induced spin connection on $\mathcal{S} \cong \det T^* \otimes |\det T|^{\frac{1}{2}}$ given by

$$D(\alpha \otimes \mu^{-\frac{1}{2}}) = \nabla(\alpha \otimes \mu^{-\frac{1}{2}}),$$

with Dirac operator

$$\not{D}(\alpha \otimes \mu^{-\frac{1}{2}}) = (d\alpha + (\mu^{\frac{1}{2}}\nabla\mu^{-\frac{1}{2}}) \wedge \alpha) \otimes \mu^{-\frac{1}{2}}.$$

Combining these formulae with (2.14), it is easy to see that \not{d}_0 coincides with the twisted de Rham differential $d - H \wedge$, as claimed in Example 2.8.

3. GENERALIZED METRICS AND TORSION-FREE CONNECTIONS

3.1. The many Levi-Civita connections. In this section we investigate an analogue in generalized geometry of the *Fundamental Lemma of pseudo-Riemannian geometry*, that is, the existence of a unique torsion-free connection compatible with a given metric. In generalized geometry there is always a freedom to choose a compatible torsion-free generalized connection, that we will constrain via the Weyl fixing mechanism introduced in Lemma 2.4. This will lead us to the definition of four canonical operators in the next sections, providing a weak analogue of the Fundamental Lemma.

Let (t, s) be the signature of the pairing on the Courant algebroid E . A generalized metric of signature (p, q) , or simply a metric on E , is a reduction of the $O(t, s)$ -bundle of frames of E to

$$O(p, q) \times O(t - p, s - q) \subset O(t, s).$$

Alternatively, it is given by an orthogonal decomposition

$$E = V_+ \oplus V_-$$

such that the restriction of the metric on E to V_+ is a non-degenerate metric of signature (p, q) . A generalized metric determines a vector bundle isomorphism

$$G: E \rightarrow E,$$

with ± 1 -eigenspace V_{\pm} , which is symmetric, $G^* = G$, and squares to the identity, $G^2 = \text{Id}$. The endomorphism G determines completely the metric, as V_+ is recovered by

$$V_+ = \text{Ker}(\text{Id} - G).$$

Definition 3.1 ([30]). A generalized connection D on E is compatible with a generalized metric $V_+ \subset E$ if

$$D(\Gamma(V_{\pm})) \subset \Gamma(E^* \otimes V_{\pm}).$$

A generalized connection D compatible with V_+ is determined by four first-order differential operators

$$\begin{aligned} D_-^+ : \Gamma(V_+) &\rightarrow \Gamma(V_-^* \otimes V_+), & D_+^- : \Gamma(V_-) &\rightarrow \Gamma(V_+^* \otimes V_-), \\ D_+^+ : \Gamma(V_+) &\rightarrow \Gamma(V_+^* \otimes V_+), & D_-^- : \Gamma(V_-) &\rightarrow \Gamma(V_-^* \otimes V_-), \end{aligned}$$

satisfying the Leibniz rule (formulated in terms of $\pi_{\pm} = \pi|_{V_{\pm}}$). Thus, the space of V_+ -compatible generalized connections on E , which we denote $\mathcal{D}(V_+)$, is an affine space modelled on

$$\Gamma(E^* \otimes \mathfrak{o}(V_+)) \oplus \Gamma(E^* \otimes \mathfrak{o}(V_-)). \quad (3.1)$$

As we shall see next, the *mixed-type operators* D_{\mp}^{\pm} are fixed, if we vary a generalized connection D inside $\mathcal{D}(V_+)$ while preserving the torsion T_D . Furthermore, when the torsion is of *pure-type*, that is

$$T_D \in \Gamma(\Lambda^3 V_+^* \oplus \Lambda^3 V_-^*),$$

then D_{\mp}^{\pm} are uniquely determined by the generalized metric and the bracket on E . Given $e \in \Gamma(E)$, we denote its orthogonal projection onto V_{\pm} by

$$e^{\pm} = \frac{1}{2}(\text{Id} \pm G)e.$$

Lemma 3.2. *Let $D \in \mathcal{D}(V_+)$ with torsion $T \in \Gamma(\Lambda^3 E^*)$.*

- i) If $D' \in \mathcal{D}(V_+)$ and $T_{D'} = T$, then $(D')_{\mp}^{\pm} = D_{\mp}^{\pm}$.*
- ii) Furthermore, if T is of pure-type then the mixed-type operators D_{\mp}^{\pm} are uniquely determined by*

$$D_{e_1^-} e_2^+ = [e_1^-, e_2^+]^+, \quad D_{e_1^+} e_2^- = [e_1^+, e_2^-]^-. \quad (3.2)$$

Proof. Given $e_1, e_2, e_3 \in \Gamma(E)$, we have

$$T(e_1^-, e_2^+, e_3^+) = (D_{e_1^-} e_2^+ - [e_1^-, e_2^+], e_3^+), \quad (3.3)$$

and therefore *ii)* follows. To prove *i)*, we subtract from (3.3) the analogue expression for $T_{D'}(e_1^-, e_2^+, e_3^+)$. \square

The second part of the previous lemma provides a weak analogue of the Koszul formula in pseudo-Riemannian geometry. The canonical mixed-type operators (3.2) were first considered by Gualtieri in the context of exact Courant algebroids [30], in order to construct a generalized geometric analogue of the Bismut connection in hermitian geometry. In our work, the main emphasis is on the space of torsion-free V_+ -compatible generalized connections, which we denote

$$\mathcal{D}^0(V_+) = \mathcal{D}(V_+) \cap \mathcal{D}^0.$$

Proposition 3.3. *Let V_+ be a generalized metric on a Courant algebroid E . Then, there exists a torsion-free generalized connection on E which is compatible with V_+ .*

Proof. We construct first a V_+ -compatible generalized connection D on E which has torsion of pure-type. For this, we define the mixed-type operators D_{\mp}^{\pm} by (3.2). To define the *pure-type operators*, we choose arbitrary metric connections ∇^+ on V_+ and ∇^- on V_- , and set

$$D_{e_1^{\pm}} e_2^{\pm} := \nabla_{\pi(e_1^{\pm})}^{\pm} e_2^{\pm}.$$

Finally, we define our torsion-free generalized connections D^0 by ‘killing the torsion’, that is (see (2.5))

$$D^0 = D - \frac{1}{3}T_D,$$

where we use the metric $\langle \cdot, \cdot \rangle$ to regard

$$T_D \in \Gamma(V_+^* \otimes \mathfrak{o}(V_+)) \oplus \Gamma(V_-^* \otimes \mathfrak{o}(V_-)).$$

Crucially, the pure-type condition on T_D implies that D^0 is V_+ -compatible. \square

Given a generalized metric, the torsion-free condition does not determine a compatible generalized connection uniquely. Unlike in pseudo-Riemannian geometry, the *many Levi-Civita generalized connections* form an affine space, modelled on the pure-type mixed symmetric 3-tensors

$$\Sigma^+ \oplus \Sigma^-$$

where

$$\Sigma^{\pm} = \Gamma(V_{\pm}^{\otimes 3}) \cap \Sigma,$$

and Σ is as in (2.6). There are canonical splittings

$$\Sigma^{\pm} = \Sigma_0^{\pm} \oplus \Gamma(V_{\pm}), \quad (3.4)$$

where the first summand corresponds to ‘trace-free’ elements, in analogy with (2.8). To see this—for V_+ , say—let r_+ denote the rank of V_+ , let $\{e_i^+\}$ be an orthogonal local frame for V_+ and let $\{\tilde{e}_i^+\} \subset \Gamma(V_+)$ be the ‘dual frame’, defined so that $\langle e_i^+, \tilde{e}_j^+ \rangle = \delta_{ij}$. Then, on V_+ the splitting (3.4) corresponds to the following direct sum decomposition

$$\chi^+ = \chi_0^+ + \chi_+^{e^+} \quad (3.5)$$

for a general element $\chi^+ \in \Gamma(V_+^* \otimes \mathfrak{o}(V_+))$, where χ_0^+ is such that

$$\sum_{i=1}^{r_+} (\chi_0^+)_{e_i^+} \tilde{e}_i^+ = 0,$$

and

$$(\chi_+^{e^+})_{e_1^+} e_2^+ = \langle e_1^+, e_2^+ \rangle e^+ - \langle e^+, e_2^+ \rangle e_1^+$$

with

$$e^+ = \frac{1}{r_+ - 1} \sum_{i=1}^{r_+} \chi_{e_i^+} \tilde{e}_i^+.$$

3.2. Canonical Dirac operators. By Lemma 3.2, the freedom in the construction of a torsion-free generalized connection compatible with V_+ corresponds to the choice of pure-type operators

$$D_+^+ : \Gamma(V_+) \rightarrow \Gamma(V_+^* \otimes V_+), \quad D_-^- : \Gamma(V_-) \rightarrow \Gamma(V_-^* \otimes V_-). \quad (3.6)$$

Using spinorial geometry, in this section we construct a pair of Dirac-type operators which are independent of choices, once we fix a divergence operator div on the Courant algebroid E (see Lemma 2.4).

Let $Cl(V_+)$ and $Cl(V_-)$ denote the bundles of Clifford algebras of V_+ and V_- , respectively. We assume that $Cl(V_\pm)$ admit irreducible Clifford modules S_\pm , that we fix in the sequel. Furthermore, we assume that the line bundles $(\det S_\pm^*)^{\frac{1}{r_{S_\pm}}}$ exist, and denote

$$\mathcal{S}_\pm = S_\pm \otimes (\det S_\pm^*)^{\frac{1}{r_{S_\pm}}}, \quad (3.7)$$

where r_{S_\pm} denotes the rank of S_\pm .

With these assumptions, similarly as in Section 2.2, a generalized connection $D \in \mathcal{D}^0(V_+)$, induces canonically a pair of Dirac-type operators

$$\not{D}^+ : \Gamma(\mathcal{S}_+) \rightarrow \Gamma(\mathcal{S}_+), \quad \not{D}^- : \Gamma(\mathcal{S}_-) \rightarrow \Gamma(\mathcal{S}_-),$$

given explicitly by (e.g. for $\alpha \in \Gamma(\mathcal{S}_+)$)

$$\not{D}^+ \alpha = \frac{1}{2} \sum_{i=1}^{r_+} \tilde{e}_i^+ \cdot D_{e_i^+} \alpha.$$

In the following lemma we prove that these natural operators are uniquely determined by the generalized metric, once we Weyl fix the generalized connections by a choice of divergence operator on E (see Definition 2.3). This provides a weak analogue of the Fundamental Lemma of Riemannian geometry, that we shall use in Section 4.2 to study the Ricci tensor in generalized geometry. Let $\text{div} : \Gamma(E) \rightarrow C^\infty(M)$ be a divergence operator on E , and consider the space of V_+ -compatible generalized connections with vanishing torsion and fixed divergence

$$\mathcal{D}(V_+, \text{div}) = \{D \in \mathcal{D}(V_+) \mid T_D = 0, \text{div}_D = \text{div}\}.$$

The existence of an element in $\mathcal{D}(V_+, \text{div})$ follows by Proposition 3.3, along the lines of the proof of Lemma 2.4. By this lemma, $\mathcal{D}(V_+, \text{div})$ is an affine space modelled on

$$\Sigma_0^+ \oplus \Sigma_0^-.$$

Lemma 3.4. *The Dirac operators \not{D}^+ and \not{D}^- are independent of the choice of $D \in \mathcal{D}(V_+, \text{div})$.*

Proof. Let $D \in \mathcal{D}(V_+, \text{div})$ and consider a torsion-free V_+ -compatible generalized connection $D' = D + \chi \in \mathcal{D}^0(V_+)$ with

$$\chi = \chi^+ + \chi^- \in \Sigma^+ \oplus \Sigma^-.$$

Using the decomposition in (3.5) there exists $e^\pm \in \Gamma(V_\pm)$ such that

$$\chi^\pm = \chi_0^\pm + \chi_\pm^{e^\pm}.$$

Then, we obtain

$$\operatorname{div}_{D'} = \operatorname{div} - (r_+ - 1)\langle e^+, \cdot \rangle - (r_- - 1)\langle e^-, \cdot \rangle$$

and therefore $\operatorname{div}_{D'} = \operatorname{div}$ implies $e^+ = e^- = 0$. Define $\sigma \in \Gamma(\Lambda^3 E^*)$ by

$$\sigma(e_1, e_2, e_3) = \langle \chi_{e_1} e_2, e_3 \rangle,$$

and note that the total skew-symmetrization $c.p.(\sigma)$ vanishes, since D' and D are torsion-free. Decomposing $\sigma = \sigma^+ + \sigma^-$ in pure-types, the result follows from

$$\not{D}'^\pm = \not{D}^\pm - \frac{1}{4}c.p.(\sigma^\pm) = \not{D}^\pm.$$

□

3.3. The case of exact Courant algebroids. In this section we discuss in detail our general construction in the special case of Example 2.5. Let E be an exact Courant algebroid over a manifold M of dimension n with Ševera class $[H] \in H^3(M, \mathbb{R})$, given by a short exact sequence

$$0 \rightarrow T^* \rightarrow E \rightarrow T \rightarrow 0. \quad (3.8)$$

Consider a generalized metric

$$E = V_+ \oplus V_-$$

of signature $(n, 0)$. Then, V_+ induces an isotropic splitting of (3.8) and an isomorphism of E with the Courant algebroid in Example 2.5, for a suitable choice of closed 3-form $H \in [H]$. In this splitting, the generalized metric takes the simple form [28]

$$V_\pm = \{X \pm g(X) : X \in T\},$$

where g is the Riemannian metric induced by the isomorphism $\pi_+ : V_+ \rightarrow T$.

Define connections on T with skew torsion, compatible with the metric g , given by

$$\nabla^\pm = \nabla^g \pm \frac{1}{2}g^{-1}H, \quad (3.9)$$

where ∇^g denotes the Levi-Civita connection of the metric g on M . As proved in [30], one has the following formula for the canonical mixed-type operators (3.2):

$$\begin{aligned} [X^-, Y^+]^+ &= \frac{1}{2}(\nabla_X^+ Y)^+, \\ [X^+, Y^-]^- &= \frac{1}{2}(\nabla_X^- Y)^-, \end{aligned} \quad (3.10)$$

where $X, Y \in \Gamma(T)$. Following the proof of Proposition 3.3, we use the isomorphisms $\pi_\pm : V_\pm \rightarrow T$ to regard ∇^\pm as metric connections on V_\pm , and define a V_+ -compatible connection with pure-type torsion. By doing so, we obtain the *Gualtieri-Bismut connection* with pure-type operators

$$\begin{aligned} D_{X^+}^B Y^+ &= \frac{1}{2}(\nabla_X^+ Y)^+, \\ D_{X^-}^B Y^- &= \frac{1}{2}(\nabla_X^- Y)^-, \end{aligned} \quad (3.11)$$

and torsion $T_{D^B} = \pi_+^* H + \pi_-^* H$. The torsion-free generalized connection $D^0 \in \mathcal{D}^0(V_+)$ constructed from D^B by

$$D^0 = D^B - \frac{1}{3}T_{D^B}$$

has pure-type operators

$$\begin{aligned} D_{X^+}^0 Y^+ &= \frac{1}{2} \left(\nabla_X^{+1/3} Y \right)^+, \\ D_{X^-}^0 Y^- &= \frac{1}{2} \left(\nabla_X^{-1/3} Y \right)^-, \end{aligned} \quad (3.12)$$

where $\nabla^{\pm 1/3}$ denote the metric connection with skew-symmetric torsion

$$\nabla^{\pm 1/3} = \nabla^g \pm \frac{1}{6} g^{-1} H. \quad (3.13)$$

Consider now the generalized connection D induced by $\nabla^g \oplus \nabla^{g*}$, as in Example 2.5. It is easy to see that D^B and D^0 differ from D by totally skew-symmetric elements in $\Gamma(E^* \otimes \mathfrak{o}(E))$ (see [21]) and therefore

$$\operatorname{div}_{D^B}(e) = \operatorname{div}_{D^0}(e) = \mu_g^{-1} L_{\pi(e)} \mu_g, \quad (3.14)$$

where $\mu_g \in \Gamma(|\det T^*|)$ is the Riemannian 1-density. Denote $\operatorname{div} = \operatorname{div}_{D^0}$ and consider

$$\operatorname{div}^\varphi = \operatorname{div} - \langle \varphi, \cdot \rangle, \quad (3.15)$$

for a 1-form $\varphi \in \Gamma(T^*)$. Then, we can construct an element $D^\varphi \in \mathcal{D}(V_+, \operatorname{div}^\varphi)$ by

$$D^\varphi = D^0 + \frac{1}{n-1} \left(\chi_+^{\varphi^+} + \chi_-^{\varphi^-} \right), \quad (3.16)$$

with pure-type operators

$$\begin{aligned} D_{X^+}^\varphi Y^+ &= \frac{1}{2} \left(\nabla_X^{+1/3} Y + \frac{1}{2(n-1)} (g(X, Y) g^{-1} \varphi - \varphi(Y) X) \right)^+, \\ D_{X^-}^\varphi Y^- &= \frac{1}{2} \left(\nabla_X^{-1/3} Y + \frac{1}{2(n-1)} (g(X, Y) g^{-1} \varphi - \varphi(Y) X) \right)^-. \end{aligned} \quad (3.17)$$

Assuming that M is a spin manifold, the canonical Dirac operators \not{D}^{φ^+} and \not{D}^{φ^-} associated to (V_+, \not{d}) in Lemma 3.4 exist, and are given by

$$\not{D}^{\varphi^\pm} = \nabla^{\pm 1/3} + \frac{1}{4} \varphi \cdot. \quad (3.18)$$

Consider now the spinor bundle $S = \Lambda^* T^*$ and the twisted spinor bundle

$$\mathcal{S} = \Lambda^* T^* \otimes |\det T|^{1/2},$$

and identify $\mathcal{S} \cong \mathbb{S}$ using $\mu_g^{\frac{1}{2}}$ (see Example 2.8). A calculation shows that

$$\not{D}^B = d - H \wedge - \frac{1}{4} T_{D^B},$$

and therefore the generalized connection D^φ defined by (3.17) satisfies

$$\not{D}^\varphi = d - H \wedge + \frac{1}{4} \varphi \wedge,$$

which in particular implies $\not{D}^0 = \not{d}_0$. Note that, by Example 2.8, \not{D}^φ is a generating operator if and only if $d\varphi = 0$.

4. THE RICCI TENSOR

4.1. Curvature and the First Bianchi Identity. Let E be Courant algebroid over a smooth manifold M . We fix a generalized metric $E = V_+ \oplus V_-$ of arbitrary signature. All the generalized connections D on E considered in this section are compatible with V_+ . Given such a generalized connection D there are two curvature operators

$$R_D^\pm \in \Gamma(V_\pm^* \otimes V_\mp^* \otimes \mathfrak{o}(V_\pm)),$$

defined by

$$\begin{aligned} R_D^+(e_1^+, e_2^-)e_3^+ &= D_{e_1^+}D_{e_2^-}e_3^+ - D_{e_2^-}D_{e_1^+}e_3^+ - D_{[e_1^+, e_2^-]}e_3^+, \\ R_D^-(e_1^-, e_2^+)e_3^- &= D_{e_1^-}D_{e_2^+}e_3^- - D_{e_2^+}D_{e_1^-}e_3^- - D_{[e_1^-, e_2^+]}e_3^-. \end{aligned} \quad (4.1)$$

The curvatures of a torsion-free generalized connection are not an invariant of the generalized metric V_+ , but rather depend on D via the choice of pure-type operators (3.6). Our next objective is to define curvature quantities which only depend on the generalized metric V_+ , after Weyl fixing. Before we address this question in Section 4.2, we prove an algebraic property of the curvatures of a torsion-free generalized connection, that we will use later in Lemma 4.7.

Proposition 4.1 (First Bianchi identity). *Let D be a torsion-free generalized connection compatible with V_+ . Then, for any $e^\pm, e_1^\pm, e_2^\pm, e_3^\pm \in \Gamma(V_\pm)$ we have*

$$c.p.123 \langle R_D^\pm(e_1^\pm, e_2^\mp)e_3^\pm, e_3^\pm \rangle = 0. \quad (4.2)$$

Proof. We argue for R_D^+ , as the other case is symmetric. Using Lemma 3.2, and properties (C4) and (C5) of the bracket (see Section 2.1)—which in particular imply $[e^-, e_i^+] = -[e_i^+, e^-]$ —the left hand side of (4.2) equals

$$\begin{aligned} & -\langle [[e_1^+, e^-], e_2^+], e_3^+ \rangle + \langle [[e_2^+, e^-], e_1^+], e_3^+ \rangle - \langle [[e_3^+, e^-], e_1^+], e_3^+ \rangle \\ & + \pi(e_2^+)(\langle [e_3^+, e^-], e_1^+ \rangle) - \pi(e_1^+)(\langle [e_3^+, e^-], e_2^+ \rangle) - \pi(e_3^+)(\langle [e_2^+, e^-], e_1^+ \rangle) \\ & - \pi(e^-)(T_D(e_1^+, e_2^+, e_3^+)) - \pi(e^-)(\langle [e_1^+, e_2^+], e_3^+ \rangle). \end{aligned}$$

Applying the condition $T_D = 0$ in the last summand, we obtain

$$c.p.123 \langle R_D^+(e_1^+, e_2^-)e_3^+, e_3^+ \rangle = \langle [[e^-, e_1^+], e_2^+] + [e_1^+, [e^-, e_2^+]] - [e^-, [e_1^+, e_2^+]], e_3^+ \rangle,$$

which vanishes by the Jacobi identity for the bracket on E (property (C1)). \square

The *algebraic Bianchi identity* (4.2) for torsion-free generalized connections is a remarkable property, as their construction typically involves standard connections with skew-torsion in the tangent bundle of M . Recall that a torsionful connection on a manifold satisfies complicated Bianchi identities which involve first-order derivatives of its torsion (see e.g. [1, Theorem 2.6]). We illustrate this with the following example.

Example 4.2. Let V_+ be a Riemannian generalized metric on an exact Courant algebroid E over M , as considered in Section 3.3. Using the isomorphisms $\pi_\pm: V_\pm \rightarrow T$ we can identify the curvatures of the Gualtieri-Bismut generalized connection with

$$R_{DB}^\pm \cong R_{\nabla^\pm},$$

where R_{∇^\pm} is the (standard) curvature of the metric connection with skew-symmetric torsion $\pm H$ in (3.9) (see [20, p. 4] for an explicit formula). Note that the first Bianchi identity for R_{∇^\pm} involves covariant derivatives of the form $\nabla^g H$, as well as quadratic terms in H (see e.g. [1]).

Considering now the torsion-free generalized connection $D^\varphi \in \mathcal{D}(V_+, \emptyset)$ defined by (3.16), its curvature is given by [21] (e.g. for $R_{D^\varphi}^+$)

$$R_{D^\varphi}^+(X^+, Y^-)Z^+ = \frac{1}{4} \left(R^{1,3}(X, Y)Z + \frac{1}{4(n-1)} (g(X, Z)\nabla_Y^+ \varphi - i_Z (\nabla_Y^+ \varphi) X) \right)^+$$

where

$$\begin{aligned} R^{1,3}(X, Y)Z &= R^g(X, Y)Z + g^{-1} \left(\frac{1}{2} (\nabla_X^g H)(Y, Z, \cdot) - \frac{1}{6} (\nabla_Y^g H)(X, Z, \cdot) \right. \\ &\quad + \frac{1}{12} H(X, g^{-1} H(Y, Z, \cdot), \cdot) - \frac{1}{12} H(Y, g^{-1} H(X, Z, \cdot), \cdot) \\ &\quad \left. - \frac{1}{6} H(Z, g^{-1} H(X, Y, \cdot), \cdot) \right). \end{aligned}$$

The tensor $R^{1,3}$ is an hybrid of the *second covariant derivatives* for ∇^+ , $\nabla^{1/3}$, and ∇^- with the remarkable property that satisfies the algebraic Bianchi identity (4.2). More invariantly, it can be written as

$$R^{1/3}(X, Y)Z = \nabla_X^{1/3} \nabla_Y^+ Z - \nabla_Y^+ \nabla_X^{1/3} Z + \nabla_{\nabla_Y^+ X}^{1/3} Z - \nabla_{\nabla_X^+ Y}^+ Z. \quad (4.3)$$

4.2. Definition of the Ricci tensor. In order to define the Ricci tensor of a generalized metric, we introduce next the definition of Ricci tensor for a metric-compatible generalized connection, due to Gualtieri (see e.g. [21, 44]).

Definition 4.3. The *Ricci tensors* $Ric_D^\pm \in \Gamma(V_\mp^* \otimes V_\pm^*)$ of a generalized connection compatible with V_+ are defined by

$$Ric_D^\pm(e^\mp, e^\pm) = \text{tr}(d^\pm \rightarrow R_D^\pm(d^\pm, e^\mp)e^\pm), \quad (4.4)$$

for $e^\pm \in \Gamma(V_\pm)$.

Our next result investigates the variation of the Ricci tensors when we deform a torsion-free generalized connection in $\mathcal{D}(V_+)$, while preserving the pure-type condition of the torsion and the divergence (see Definition 2.3).

Proposition 4.4. *Let $D, D' \in \mathcal{D}(V_+)$ be generalized connections with the same divergence $\text{div}_D = \text{div}_{D'}$. If $T_D = 0$ and $T_{D'}$ is of pure-type then $Ric_D^\pm = Ric_{D'}^\pm$.*

Proof. Setting $\chi = D' - D$, the condition $\text{div}_D = \text{div}_{D'}$ combined with $T_{D'}$ being of pure-type imply that (see (3.4))

$$\chi = \chi^+ + \chi^- \in \Sigma_0^+ \oplus \Sigma_0^-.$$

Given $e^\pm \in \Gamma(V_\pm)$ we calculate

$$\begin{aligned}
Ric_D^+(e^-, e^+) &= \sum_{i=1}^{r_+} \langle \tilde{e}_i^+, R_{D'}^+(e_i^+, e^-) e^+ \rangle \\
&= Ric_D^+(e^-, e^+) + \sum_{i=1}^{r_+} \langle \tilde{e}_i^+, \chi_{e_i^+}^+(D_e^- e^+) - D_e^-(\chi_{e_i^+}^+ e^+) - \chi_{[e_i^+, e^-]^+}^+ e^+ \rangle \\
&= Ric_D^+(e^-, e^+) + \sum_{i=1}^{r_+} \left(-\langle \chi_{e_i^+}^+ \tilde{e}_i^+, D_e^- e^+ \rangle + \langle \chi_{e_i^+}^+ e^+, D_e^- \tilde{e}_i^+ \rangle \right. \\
&\quad \left. + \iota_{\pi(e_-)} d \langle \chi_{e_i^+}^+ \tilde{e}_i^+, e^+ \rangle - \langle \chi_{[e^-, e_i^+]^+}^+ \tilde{e}_i^+, e^+ \rangle \right) \\
&= Ric_D^+(e^-, e^+) + \sum_{i=1}^{r_+} \left(\langle \chi_{e_i^+}^+ e^+, D_e^- \tilde{e}_i^+ \rangle - \langle \chi_{[e^-, e_i^+]^+}^+ \tilde{e}_i^+, e^+ \rangle \right),
\end{aligned}$$

where we have used $\sum_{i=1}^{r_+} \chi_{e_i^+}^+ \tilde{e}_i^+ = 0$. To conclude, we construct a special orthogonal frame in order to evaluate the last expression. Given $x \in M$ and $e_x^- \in V_-$, we choose an orthogonal frame $\{e_i^+\}$ around x for V_+ which satisfies $D_{e_x^-} e_i^+ = 0$ at x . This frame can be constructed by smooth extension of the parallel transport for the V_- -connection D_-^+ along a curve starting at x with initial velocity $\pi(e_x^-)$ (see e.g. [38]). The proof now follows from the previous formula. \square

Given a divergence operator div on the Courant algebroid E , Lemma 2.4 combined with Proposition 4.4 implies that $Ric_D^\pm = Ric_{D'}^\pm$ for any pair $D, D' \in \mathcal{D}(V_+, div)$. This leads us to our definition of the Ricci tensor.

Definition 4.5. Let div be a divergence operator on the Courant algebroid E , and let $V_+ \subset E$ be a generalized metric on E . The *Ricci tensors*

$$Ric^\pm \in \Gamma(V_\mp^* \otimes V_\pm^*)$$

of the pair (V_+, div) are defined by $Ric^\pm = Ric_D^\pm$ for any choice of generalized connection $D \in \mathcal{D}(V_+, div)$.

Remark 4.6. By Proposition 4.4, the Ricci tensor Ric_D^\pm is independent of the choice of D , among generalized connections compatible with V_+ , with pure-type torsion and fixed divergence $div_D = div$.

We continue our discussion with an alternative point of view on the Ricci tensor, using the canonical Dirac operators constructed in Section 3.2. For this, we assume that V_\pm admit spinor bundles S_\pm and consider the twisted spinor bundles \mathcal{S}_\pm endowed with the canonical Dirac operators \not{D}^\pm on $\Gamma(\mathcal{S}_\pm)$ as in Lemma 3.4.

Lemma 4.7. Let $V_+ \subset E$ be a generalized metric and let div be a divergence operator on E . Then, for any choice of $D \in \mathcal{D}(V_+, div)$ the Ricci tensors in Definition 4.5 can be calculated by

$$\iota_{e^\mp} Ric^\pm \cdot \alpha = 4(\not{D}^\pm D_{e^\mp} - D_{e^\mp} \not{D}^\pm) \alpha, \quad (4.5)$$

where $\alpha \in \Gamma(\mathcal{S}_\pm)$ and $e^\mp \in \Gamma(V_\mp)$.

We note that the right hand side of (4.5) is not tensorial in $e^\mp \in \Gamma(V_\mp)$, and therefore it should be understood as follows (e.g. for Ric^+): given $x \in M$ and $e_x^- \in V_-$, we choose an orthogonal frame $\{e_i^+\}$ around x for V_+ which satisfies $D_{e_x^-} e_i^+ = 0$ at x . This frame can be constructed as in the proof of Proposition 4.4,

by smooth extension of the parallel transport for the V_- -connection D_-^+ along a curve starting at x with initial velocity $\pi(e_x^-)$ (see e.g. [38]). Then, using the V_+ -connection D_+^+ , extend e_x^- by parallel transport along curves γ_i starting at x with velocities $\pi(e_{i|x}^+) \in T_x M$. Finally, take the following expression as the definition of the right hand side of (4.5) at the point $x \in M$:

$$2 \sum_{i=1}^{r_+} \tilde{e}_i^+ \cdot D_{e_i^+} (D_{e^-} \alpha) - 2 D_{e^-} \sum_{i=1}^{r_+} (\tilde{e}_i^+ \cdot D_{e_i^+} \alpha). \quad (4.6)$$

Even though the operators \mathcal{D}^+ and D_-^+ involved in (4.6) are independent of D , the parallel transport of e_x^- along the curves γ_i may depend on the choice of pure-type operator D_+^+ . Thus, in principle formula (4.5) cannot be taken as a definition of the Ricci tensor, without relying on Proposition 4.4.

Proof of Lemma 4.7. Using (4.6) as the definition of the right hand side of (4.5) at $x \in M$, with the conventions above we have

$$\begin{aligned} ([\mathcal{D}^\pm, D_{e^\mp}] \alpha)|_x &= \frac{1}{2} \sum_{i=1}^{r_+} \tilde{e}_i^+ \cdot (D_{e_i^+} D_{e^-} - D_{e^-} D_{e_i^+} - D_{[e_i^+, e^-]}) \alpha \\ &= \frac{1}{2} \sum_{i=1}^{r_+} \tilde{e}_i^+ \cdot R_D^+(e_i^+, e^-) \cdot \alpha \\ &= \frac{1}{4} \text{tr}(d^+ \rightarrow R_D^+(d^+, e^-)) \cdot \alpha \\ &\quad - \frac{1}{4} \sum_{i < j < k} \left(c \cdot p \cdot ijk \langle R_D^+(e_i^+, e^-) e_j^+, e_k^+ \rangle \right) e_i^+ \cdot e_j^+ \cdot e_k^+ \cdot \alpha, \end{aligned}$$

where in the first equality we have used that $[e^-, e_i^+] = D_{e^-} e_i^+$ by Lemma 3.2, and $(D_{e^-} e_i^+)|_x = 0$ by the choice of frame. The proof follows from the algebraic Bianchi identity in Proposition 4.1. \square

Remark 4.8. Our formula (4.5) is an interpretation of a formula for the Ricci tensor of a generalized metric in the physics literature [15], claimed without proof. Our proof relies in the Bianchi identity (4.1) for the generalized curvature of a torsion free-generalized connection, which seems to be new.

We finish this section with an explicit calculation of the Ricci tensors in an example. Further explicit calculations of Ricci tensors on transitive Courant algebroids are provided in Section 7.

Example 4.9. From the formulae in Example 4.2, we obtain the following expression for the Ricci tensors of $(V_+, \text{div}^\varphi)$ (see (3.15))

$$\text{Ric}^\pm = \text{Ric}_g - \frac{1}{4} H \circ H \mp \frac{1}{2} d^* H - \frac{1}{4} \nabla^\pm \varphi, \quad (4.7)$$

where $H \circ H$ denotes the symmetric tensor $(H \circ H)_{ij} = g^{rs} g^{kl} H_{irk} H_{jsl}$. We notice an important property of these tensors—strongly reminiscent of the symmetric property of the Ricci tensor in Riemannian geometry—for which we have no Lie theoretic proof: the decomposition $\text{Ric}^\pm = h^\pm + b^\pm$ into symmetric and skew symmetric parts, respectively, satisfies:

$$h^+ = h^-, \quad b^+ = -b^-. \quad (4.8)$$

Some consequences of this important symmetry will be discussed in Section 5. Even though this property is lost when we deform div^φ away from the kernel of the anchor map, we note that deformations by closed 1-forms are the most natural ones, as they correspond to deformations of a Dirac generating operator on the exact Courant algebroid (see Section 3.3). When $\varphi = 0$, Ric^\pm equals the Ricci tensor of the connection ∇^\pm , which corresponds to the Ricci tensor for the Gualtieri-Bismut connection Ric_{DB}^\pm . We note here that $\text{div}_{DB} = \text{div}_{D^0}$, in agreement with Proposition 4.4.

5. RICCI FLOW AND THE KILLING SPINOR EQUATIONS

5.1. The Ricci flow in generalized geometry. Let E be a Courant algebroid over a smooth manifold M . Throughout this section, we consider the alternative definition of a generalized metric given in terms of an orthogonal endomorphism

$$G: E \rightarrow E$$

inducing a non-degenerate pairing $\langle G \cdot, \cdot \rangle$ and squaring to the identity $G^2 = \text{Id}$. With this definition, the orthogonal decomposition $E = V_+ \oplus V_-$ corresponds to the ± 1 -eigenbundle decomposition of E with respect to G .

Given a smooth one-parameter family of generalized metrics G_t , the variation $\partial_t G$ is an skew-symmetric endomorphisms of E , that is

$$\partial_t G \in \Gamma(\mathfrak{o}(E)),$$

which exchanges the subbundles V_+ and V_- , yielding a pair of endomorphisms

$$\partial_t G^+: V_- \rightarrow V_+,$$

$$\partial_t G^-: V_+ \rightarrow V_-.$$

We are ready to introduce our notion of *Ricci flow*.

Definition 5.1. Let E be a Courant algebroid over a smooth manifold M . We say that a one-parameter family (G_t, div_t) of generalized metrics G_t and divergence operators div_t is a solution of the *Ricci flow* if

$$\partial_t G^+ = -2\text{Ric}_t^+, \tag{5.1}$$

where Ric_t^+ is the Ricci tensor of (G_t, div_t) as in Definition 4.5.

In the previous definition, the Ricci tensors Ric^\pm of (G, div) are regarded as endomorphisms

$$\text{Ric}^\pm: V_\mp \rightarrow V_\pm,$$

using the induced metric on V_\pm to identify $V_\pm \cong V_\pm^*$. Note that the choice of Ric^+ instead of Ric^- is just a matter of convention, since we are considering metrics with arbitrary signature.

We show next that the Ricci flow introduced in Definition 5.1 extends a previous notion of the Ricci flow on exact Courant algebroids by Gualtieri (see [44]). To see this, we assemble the Ricci tensors Ric^\pm into a unique endomorphism

$$P = \frac{1}{2}(\text{Ric}^+ - \text{Ric}^-) \in \Gamma(\text{End } E)$$

and consider the evolution equation

$$\partial_t G = -2[P_t, G_t]. \tag{5.2}$$

When E is an exact Courant algebroid and $\partial_t \text{div} = 0$, (5.2) coincides with Gualtieri's definition for the *generalized Ricci flow*. Note that the right hand side of this equation is $-2(\text{Ric}_t^+ + \text{Ric}_t^-)$, and therefore (5.2) is equivalent to the system

$$\begin{aligned}\partial_t G^+ &= -2\text{Ric}_t^+, \\ \partial_t G^- &= -2\text{Ric}_t^-. \end{aligned} \tag{5.3}$$

A priori, these equations are stronger than the Ricci flow (5.1). The equivalence between the flows (5.1) and (5.2) relies on an expected symmetry of the Ricci tensor in Definition 4.5, for which we still do not have a general proof. To motivate the next definition, note that if (G_t, div_t) is a solution of the system (5.3), then the *total Ricci tensor* given by

$$\text{Ric} := \text{Ric}^+ + \text{Ric}^- \in \Gamma(\text{End } E)$$

must be an element in $\Gamma(\mathfrak{o}(E))$, as so is $\partial_t G$.

Definition 5.2. A pair (V_+, div) has the *skew-symmetry property* if the total Ricci tensor $\text{Ric} := \text{Ric}^+ + \text{Ric}^-$ is skew-orthogonal with respect to the ambient metric on E , that is,

$$\text{Ric} \in \Gamma(\mathfrak{o}(E)).$$

On an exact Courant algebroid, pairs $(V_+, \text{div}^\varphi)$ given by a generalized metric $V_+ \subset E$ such that $V_+ \cap T^* = \{0\}$ and a divergence operator div^φ as in (3.15) have the skew-symmetry property. Note that $V_+ \cap T^* = \{0\}$ is the condition for V_+ to induce a metric g on M . Then, our claim follows by the explicit calculation in Example 4.9, and the symmetry (4.8) between the symmetric and skew-symmetric parts of the Ricci tensors. If we deform φ away from $\text{Ker } \pi$, then it is easy to see that the skew-symmetry property is lost (note from Example 2.8 that such a deformation does not correspond to a deformation of Dirac generating operators). More generally, a similar result is true for transitive Courant algebroids obtained by reduction (see Proposition 7.2). On general grounds, we expect any reasonable class of generalized metrics and divergence operators on a Courant algebroid to have this property, but we have not been able to give a Lie-theoretic proof of this fact.

The following result, which proves the equivalence between the two notions of Ricci flow on exact Courant algebroids, is a straightforward consequence of Definition 5.2.

Proposition 5.3. *A solution (G_t, div_t) of the Ricci flow (5.1) is also a solution of the generalized Ricci flow (5.2) if and only if (G_t, div_t) has the skew-symmetry property for all t .*

A general discussion about existence results for the generalized Ricci flow on exact Courant algebroids can be found in [44].

Example 5.4. Let (G_t, div_t) be a solution of the generalized Ricci flow (5.2) on an exact Courant algebroid E , with G_t inducing a metric g_t on M for all t . Fixing an isotropic splitting for E , the family (G_t, div_t) can be written as

$$G_t = e^{b_t} \begin{pmatrix} 0 & g_t^{-1} \\ g_t & 0 \end{pmatrix} e^{-b_t}$$

$$\text{div}_t(e) = \mu_{g_t}^{-1} L_{\pi(e)} \mu_{g_t} - \langle e_t, e \rangle$$

where b_t is the 2-form which codifies the isotropic splitting determined by G_t , H is the closed 3-form in the fixed isotropic splitting and $e_t \in \Gamma(E)$. For simplicity,

we consider that $e_t = \varphi_t \in \Gamma(T^*)$ is a family of 1-forms on M . Without loss of generality, we assume $b_{t_0} = 0$, and then the derivative of G_t at times t_0 is

$$\partial_t G = [\partial_t b, G_{t_0}] + \begin{pmatrix} 0 & -g_t^{-1} \partial_t g_t g_t^{-1} \\ \partial_t g_t & 0 \end{pmatrix}$$

which implies that the system (5.3) is equivalent to (see (4.7))

$$\begin{aligned} \partial_t g_t - \partial_t b_t &= -2Ric^+ = -2(Ric_{g_t} - \frac{1}{4}H_t \circ H_t - \frac{1}{2}d^*H_t - \frac{1}{4}\nabla^+\varphi_t), \\ \partial_t g_t + \partial_t b_t &= -2Ric^- = -2(Ric_{g_t} - \frac{1}{4}H_t \circ H_t + \frac{1}{2}d^*H_t - \frac{1}{4}\nabla^-\varphi_t). \end{aligned} \quad (5.4)$$

These two equations are equivalent by the symmetry (4.8). Furthermore, the first equation is (literally) the Ricci flow of (G_t, div_t) in Definition 5.1.

Remark 5.5. In the case that $\varphi_t = 2d\phi_t$ is exact, the equations (5.4) corresponds to the renormalization group flow in Type II string theory [11] (see also [17]), and therefore have an important physical motivation. In this context, the symmetric and skew-symmetric parts of the Ricci tensor Ric^+ are (essentially) the β -functions for the gravitational field and the B -field, respectively, which control the conformal invariance of the effective action in the sigma model approach to the theory. The function potential ϕ_t is the so-called *dilaton field*, which plays an important role in T-duality (see Section 6.2).

5.2. Killing spinors in generalized geometry. The stationary points of the Ricci flow (5.1) correspond to *Ricci flat pairs* (V_+, div) , that is, satisfying

$$Ric^+ = 0. \quad (5.5)$$

Being the Ricci flat condition a second order equation, it is natural to ask whether it can be regarded as an integrability condition for a first-order equation. This motivates our next definition.

Definition 5.6. We say that a pair (V_+, div) is a solution of the *Killing spinor equations*, if there exists a real spinor bundle S_+ for V_+ and a non-vanishing section of the twisted spinor bundle $\eta \in \Gamma(S_+)$ such that

$$\begin{aligned} D_-^+ \eta &= 0, \\ \not{D}^+ \eta &= 0, \end{aligned} \quad (5.6)$$

where D_-^+ and \not{D}^+ are the operators defined in Lemma 3.2 and Lemma 3.4.

Lemma 3.2 and Lemma 3.4 imply that these equations are independent of the choice of torsion-free generalized connection, as the operators D_-^+ and \not{D}^+ are canonically defined by the pair (V_+, div) . Definition 5.6 generalizes [23, Definition 5.3] to arbitrary Courant algebroids, and it is inspired by the approach to the *Killing spinor equations in supergravity* in [15, 25] (see Remark 5.10 and Remark 5.9).

In the next proposition we state the precise relation between solutions of the Killing spinor equations and Ricci flat pairs. This result, that follows as a direct consequence of Lemma 4.7, can be seen as an analogue of a classical result in Riemannian geometry (resp. pseudo-Riemannian geometry): metrics with parallel spinor fields are Ricci flat (resp. *Ricci null*).

Proposition 5.7. *If (V_+, div) is a solution of the Killing spinor equations (5.6), then the image of the endomorphism $Ric^+ : V_- \rightarrow V_+$ consists of null vectors. In particular, if V_+ is positive definite then $Ric^+ = 0$.*

Proof. By Lemma 4.7 we have $\iota_{e_-} Ric^+ \cdot \eta = 0$. Since η is non-vanishing, Clifford multiplication with $\iota_{e_-} Ric^+$ yields $|\iota_{e_-} Ric^+|^2 = 0$, and the result follows. \square

Solutions of the Killing spinor equations (5.6) are very interesting objects, both from a geometric and from a physical point of view. To see this, let us focus on the case of a *Riemannian generalized metric* V_+ on an exact Courant algebroid E over M , that is, a generalized metric inducing a Riemannian metric on M via $\pi_+ : V_+ \rightarrow T$. In this context, the condition that V_+ admits an irreducible Clifford module in Definition 5.6 implies that M admits a spin structure. Using the anchor map and the explicit formulae (3.10) and (3.18), the Killing spinor equations (5.6) read

$$\begin{aligned} \nabla^+ \eta &= 0, \\ \nabla^{\frac{1}{3}} \eta + \frac{1}{4} \varphi \cdot \eta &= 0. \end{aligned} \tag{5.7}$$

The equations (5.7) are equivalent, in low dimensions, to the *Killing spinor equations in supergravity* [20] (see Remark 5.10 and Remark 5.9). In the case $\varphi = 0$, Proposition 5.7 combined with (4.7) imply that any solution of (5.7) is an *Einstein metric with skew torsion*, as studied in [3].

Using the explicit formula (5.7) combined with Proposition 5.7, we show next that solutions of the Killing spinor equations correspond to (standard) *special holonomy metrics* on a compact manifold, provided that φ is exact. Several proofs of this fact by different authors can be found in the mathematical physics literature (see e.g. [35]), but we have included a (dimension independent) proof to illustrate our framework. The novelty here is that metrics with special holonomy are natural objects in generalized geometry, something which does not seem to have been pointed out before (see [13, 21] for the cases of $SU(3)$ and G_2 metrics).

Proposition 5.8. *Let E be an exact Courant algebroid over a smooth compact spin manifold M . Then, a solution $(V_+, \text{div}^\varphi)$ of the Killing spinor equations (5.6) with φ exact is equivalent to a Riemannian metric with a parallel spinor.*

Proof. The proof follows by explicit calculations. Denote by g and H the Riemannian metric and the closed 3-form in the splitting $E \cong T \oplus T^*$ induced by V_+ . By assumption $\varphi = 2d\phi$ in (5.6), and by Proposition 5.7 $(V_+, \text{div}^\varphi)$ is Ricci flat (5.5). Contracting with g in the formula for Ric^+ (4.7) we obtain

$$S_g - \frac{3}{2}|H|^2 + \frac{1}{2}\Delta_g \phi = 0,$$

where we are using the conventions $|H|^2 = g^{ij}g^{kl}g^{st}H_{iks}H_{jlt}/6$ and $\Delta_g = d^*d$. Using now the Killing spinor equation (5.7) and Bismut's Lichnerowicz-type formula for the cubic Dirac operator [6] (see also [2, Theorem 6.2], and note the different sign convention for the Clifford algebra)

$$0 = 4(\nabla^{\frac{1}{3}} + \varphi/4)^2 + 4(\nabla^+)^* \nabla^+ = -S_g - 2\Delta_g \phi + |d\phi|^2 + \frac{1}{2}|H|^2.$$

Adding the two scalar equations we obtain

$$|H|^2 + \frac{3}{2}\Delta_g \phi - |d\phi|^2 = 0.$$

However, for any $\alpha \in \mathbb{R}$, and for $\Delta = d^*d$,

$$\Delta_g(e^{\alpha\phi}) = -\alpha^2 e^{\alpha\phi} |d\phi|^2 + \alpha e^{\alpha\phi} \Delta_g \phi,$$

and so

$$3\Delta(e^{\alpha\phi}) + \alpha(3\alpha - 2)e^{\alpha\phi}|d\phi|^2 + 2\alpha e^{\alpha\phi}|H|^2 = 0.$$

We can take $\alpha = 1$ and integrate over M to conclude that $H = 0$ and $d\phi = 0$. Finally, using the first equation in (5.7) it follows that $\nabla^g\eta = 0$. The converse is trivial, by taking the standard Courant algebroid $E = T \oplus T^*$ with $H = 0$ and div the Riemannian divergence (see (3.14)). \square

An alternative treatment of special holonomy metrics using generalized geometry can be found in the work of Witt [47], via the study of diagonal $G \subset G \times G \subset O(n, n)$ structures on exact Courant algebroids. We note however that an arbitrary Courant algebroid isomorphism may destroy the diagonal property, producing a generalized $G \times G$ -structure. To the knowledge of the author, the Killing spinor equations (5.7) provide the first known ‘embedding’ of special holonomy metrics onto generalized geometry. We will see some potential applications of this result in the next section.

In the case that φ is closed but not exact (see Example 2.8) there are interesting solutions of the Killing spinor equations (5.6) in exact Courant algebroids, with $H \neq 0$ and $\varphi \neq 0$. An example has been provided in ongoing joint work of the author with Shahbazi [24], where it is proved that the Boothby hermitian metric in a complex Hopf surface is a solution. In this case, the 1-form φ in (3.12) is given by the closed Lee form of the hermitian structure.

More generally, for transitive Courant algebroids obtained by reduction, it has been proved in [23] that the system (5.6) corresponds (essentially) to the Strominger system of partial differential equations. We postpone any further explanation about this interesting subject to Section 7, where we will study the behaviour of the equations under T-duality.

Remark 5.9. In low dimensions, the equations (5.7) are equivalent to the *killing-spinor equations* in compactifications of the common sector of ten-dimensional supergravity [20], which may serve as motivation for our name. This particular way of writing the equations—which is crucial for the relation with generalized geometry—was first considered in [25] in the context of type II supergravity (in order to obtain an expression which is half-independent of the Ramond-Ramond fields).

Remark 5.10. When E is exact, a killing spinor in the sense of Definition 5.6 is a hybrid between the notions of parallel spinor and harmonic spinor for connections with skew-symmetric torsion. The name *Killing spinor* for a solution of the equations (5.6) may be misleading, as there is a well-established notion of Killing spinor in pseudo-Riemannian geometry (see [1] for a review on these topics). The alternative given by *generalized Killing spinor* does not do any better, according to the classical notion by Friedrich, Kim, Bär, Gauduchon, and Moroianu. Despite this, in the most interesting examples (see Section 7) the equations (5.6) are motivated by physics and have a well-established name.

6. T-DUALITY IN GENERALIZED GEOMETRY

6.1. T-duality and main result. We go now for the mainstream of our development. In this section we investigate a method to produce new solutions of the partial differential equations introduced in Section 5 based on T-duality. The idea has its origins in the string theory literature in the work of Buscher [9, 10], and was developed further by Roček and Verlinde [39]. Our main theorem builds on an important result by Cavalcanti and Gualtieri [12], which states that *topological T-duality* for principal torus bundles—as defined by Bouwknegt, Evslin and Mathai

[7]—induces suitable Courant algebroid isomorphisms, that we shall call *dualities* in this paper.

We start introducing the notion of T-duality which we shall consider. Our Definition 6.1 below is a straightforward generalization of the main implication of [12, Theorem 3.1]. Let E be a Courant algebroid over a smooth manifold M , with anchor $\pi_E: E \rightarrow TM$. Assume that a torus T acts on M freely and properly, so that M is a principal T -bundle over $M/T = B$, and that the action lifts to E preserving the Courant algebroid structure. Then, we say that (E, M, T) is an *equivariant Courant algebroid* with base B . The *simple reduction* of E by T , which we denote

$$E/T \rightarrow B,$$

is a Courant algebroid over B , with anchor $\pi_{E/T}: E/T \rightarrow TB$, whose sheaf of sections is given by the invariant section of E , that is, $\Gamma(E/T) = \Gamma(E)^T$.

Definition 6.1. Let (E, M, T) and $(\hat{E}, \hat{M}, \hat{T})$ be equivariant Courant algebroids over the same base $M/T = B = \hat{M}/\hat{T}$. We say that (E, M, T) is dual to $(\hat{E}, \hat{M}, \hat{T})$ if there exists an isomorphism of Courant algebroids between the simple reductions

$$\psi: E/T \rightarrow \hat{E}/\hat{T}.$$

In this situation, we will call ψ the *duality isomorphism*.

The aim of this section is to understand the interplay between T-duality, in the sense of the previous definition, and the existence of solutions of the Ricci flow and the Killing spinor equations introduced in Section 5. For this, we need to introduce a notion of duality for pairs (V_+, div) , given by a generalized metric and a divergence operator.

Given an equivariant Courant algebroid (E, M, T) and a T -invariant generalized metrics V_+ on E , we can push-forward V_+ along the bundle projection $p: M \rightarrow B$ to obtain a generalized metric with the same signature on the simple reduction

$$p_*V_+ \subset E/T.$$

This way we obtain an identification between T -invariant generalized metrics on E and generalized metrics on E/T . Similarly, given a T -invariant divergence operator $\text{div}: \Gamma(E) \rightarrow C^\infty(M)$ it induces a first-order differential operator

$$p_*\text{div}: \Gamma(E)^T = \Gamma(E/T) \rightarrow C^\infty(M)^T = C^\infty(B).$$

The π_E -Leibniz rule for div restricted to $p^*C^\infty(B)$ (see (2.12)), is precisely the $\pi_{E/T}$ -Leibniz rule for $p_*\text{div}$, which thus defines a divergence operator on E/T .

Definition 6.2. Let (E, M, T) and $(\hat{E}, \hat{M}, \hat{T})$ be dual equivariant Courant algebroids in the sense of Definition 6.1. We will say that a T -invariant pair (V_+, div) on E is dual to a \hat{T} -invariant pair $(\hat{V}_+, \hat{\text{div}})$ on \hat{E} if

$$\psi(p_*V_+) = \hat{p}_*\hat{V}_+ \quad \text{and} \quad p_*\text{div} = \psi^*\hat{p}_*\hat{\text{div}},$$

where $p: M \rightarrow B$ and $\hat{p}: \hat{M} \rightarrow B$ are the bundle projections.

The next result shows that the Ricci tensor in Definition 4.5 is naturally exchanged under T-duality.

Proposition 6.3. Let (E, M, T) and $(\hat{E}, \hat{M}, \hat{T})$ be equivariant dual Courant algebroids, endowed with dual invariant pairs (V_+, div) on E and $(\hat{V}_+, \hat{\text{div}})$ on \hat{E} , in the sense of Definition 6.2. Then

$$p_*\text{Ric}^\pm(V_+, \text{div}) = \psi^*\hat{p}_*\text{Ric}^\pm(\hat{V}_+, \hat{\text{div}}), \quad (6.1)$$

where Ric^\pm are as in Definition 4.5.

The proof of Proposition 6.3 requires to understand the relation between T -invariant generalized connections on E and generalized connections on E/T . Denote by $\mathcal{D}_E^T \subset \mathcal{D}_E$ the space of T -invariant generalized connections on E and by $\mathcal{D}_{E/T}$ the space of generalized connections on the simple reduction E/T .

Lemma 6.4. *There is a canonical identification $\mathcal{D}_E^T = \mathcal{D}_{E/T}$. In particular, \mathcal{D}_E^T is non-empty.*

Proof. The space $\mathcal{D}_{E/T}$ is non-empty, since we can construct a generalized connection \check{D} out of an orthogonal connection $\nabla^{E/T}$ on E/T by

$$\check{D}_{e_1} e_2 = \nabla_{\pi_{E/T}(e_1)}^{E/T} e_2.$$

The affine spaces \mathcal{D}_E^T and $\mathcal{D}_{E/T}$ are both modelled on $\Gamma(E^* \otimes \mathfrak{o}(E))^T$, via the identification $\Gamma(E/T) = \Gamma(E)^T$, and therefore it is enough to prove that a generalized connection on E/T induces an invariant connection on E . For this, we note that E is locally generated by $\Gamma(E)^T$, and therefore given $\check{D} \in \mathcal{D}_{E/T}$ we can define a generalized connection D on E by extending its action from $\Gamma(E)^T$ to $\Gamma(E)$ imposing the Leibniz rule with respect to the anchor π_E . The generalized connection D is compatible with the pairing, as for $f \in C^\infty(M)$ and $e_1, e_2 \in \Gamma(E)^T$ we have

$$\begin{aligned} \pi_E(e_1) \langle f e_2, e_3 \rangle &= df(\pi_E(e_1)) \langle e_2, e_3 \rangle + f \pi_{E/T}(e_1) (\langle e_2, e_3 \rangle) \\ &= \langle df(\pi_E(e_1)) + f \check{D}_{e_1} e_2, e_3 \rangle + f \langle e_2, \check{D}_{e_1} e_3 \rangle \\ &= \langle D_{e_1}(f e_2), e_3 \rangle + \langle e_2, D_{e_1} e_3 \rangle. \end{aligned}$$

For the first equality we have used that $\langle e_2, e_3 \rangle$ is the pull-back of a function on B and consequently

$$\pi_E(e_1) \langle e_2, e_3 \rangle = \pi_{E/T}(e_1) \langle e_2, e_3 \rangle.$$

□

We are ready for the proof of Proposition 6.3.

Proof of Proposition 6.3. Given a T -invariant divergence operator div and a T -invariant torsion-free generalize connection D on E compatible with V_+ (see Proposition 3.3), we have

$$div - div_D = \langle e', \cdot \rangle \in \Gamma(E^*)^T.$$

Considering $D' = D + \chi_+^{e^+} + \chi_-^{e^-}$ with $\chi_\pm^{e^\pm}$ as in (3.5) and setting

$$(e')^\pm = (r_\pm - 1)e^\pm,$$

we obtain $div_{D'} = div$. Thus, the affine subspace $\mathcal{D}(V_+, div)^T \subset \mathcal{D}(V_+, div)$ of T -invariant generalized connections is non-empty. Furthermore, by Lemma 6.4 there is a canonical identification

$$\mathcal{D}(V_+, div)^T = \mathcal{D}(p_* V_+, p_* div).$$

Applying Lemma 4.4 and Proposition 2.10, this implies

$$p_* Ric^\pm(V_+, div) = Ric^\pm(p_* V_+, p_* div).$$

The result follows now from

$$\{\psi_* D \mid D \in \mathcal{D}(V_+, div)^T\} = \mathcal{D}(\hat{p}_* \hat{V}_+, \hat{p}_* \hat{div}) \quad (6.2)$$

which implies

$$\psi^* Ric^\pm(\hat{p}_* \hat{V}_+, \hat{p}_* \hat{div}) = Ric^\pm(p_* V_+, p_* div).$$

□

We state next our main result, which follows as a consequence of the proof of Proposition 6.3 and equation (6.1).

Theorem 6.5. *Let (E, M, T) and $(\hat{E}, \hat{M}, \hat{T})$ be equivariant dual Courant algebroids. Then:*

- i) *If (G_t, div_t) is an invariant solution of the Ricci flow (5.1) on E , and $(\hat{G}_t, \hat{\text{div}}_t)$ is a family of invariant dual pairs on \hat{E} , then $(\hat{G}_t, \hat{\text{div}}_t)$ is also a solution of the Ricci flow.*
- ii) *If (G, div) is an invariant solution of the Killing spinor equations (5.6) on E , and $(\hat{G}, \hat{\text{div}})$ is an invariant dual pair on \hat{E} , then $(\hat{G}, \hat{\text{div}})$ is also a solution of the Killing spinor equations.*

Proof. The first part of the statement is a direct consequence of equation (6.1). As for the second part, if (V_+, div) is an invariant solution of the Killing spinor equations (5.6) then there exists an equivariant twisted spinor bundle \mathcal{S}_+ and a non-vanishing $\eta \in \Gamma(\mathcal{S}_+)^T$ satisfying (5.6). The canonical operators D_-^+ and \not{D}^+ are constructed from an arbitrary choice of element $D \in \mathcal{D}(V_+, \text{div})$, that we can take to be T -invariant by Lemma 6.4 and the proof of Proposition 6.3, and thus these operators induce canonical operators $p_*D_-^+$ and $p_*\not{D}^+$ on $\Gamma(\mathcal{S}_+/T)$. Consequently, $p_*\eta \in \Gamma(\mathcal{S}_+/T)$ satisfy (5.6) and $(p_*V_+, p_*\text{div})$ is a solution of the Killing spinor equations on E/T . Finally, using the isomorphism ψ in Definition 6.1, we can regard \mathcal{S}_+/T as a spinor bundle for $\hat{p}_*\hat{V}_+$ and η as a non-vanishing section. Since the spaces of generalized connections in (6.2) are exchanged under the isomorphism ψ , so are the operators $p_*D_-^+$ and $p_*\not{D}^+$. Therefore, $\mathcal{D}(\hat{p}_*\hat{V}_+, \hat{p}_*\hat{\text{div}})$ is also a solution of the Killing spinor equations. □

A direct consequence of the previous theorem is that the stationary points of the flow, given by solutions of the Ricci-flat equation $\text{Ric}^+ = 0$, are exchanged under T-duality. A proof of this fact for transitive Courant algebroid obtained by reduction was provided in [5], using the explicit form of the equations in this case (see [21] and Lemma (7.1)). A proof of part i) of Theorem 6.5 for exact Courant algebroids was provided in [44], using an ad hoc definition of the Ricci tensor and explicit calculations.

We should mention that our proof also works when T and \hat{T} are substituted by arbitrary non-abelian groups, but we have not been able to find any interesting examples in this case. Based on our proof, we believe that Theorem 6.6 extends to a fairly general class of Poisson-Lie T-duals, in the sense of Klimčík and Ševera [36]. We thank Ševera for clarifications about the non-abelian setup.

It is interesting to notice that Proposition 5.8 combined with Theorem 6.5 implies that special holonomy metrics with a continuous abelian group of isometries are preserved by T-duality, in the sense of Definition 6.1 (cf. [32, 46]), and we believe that this is also true for the non-abelian Poisson-Lie T-duality. Even though compact manifolds with holonomy $SU(n)$, G_2 or $Spin(7)$ have no continuous symmetries, it would be interesting to explore this perspective of the present work in the abundant local examples that exists in the literature (see e.g. [41]).

6.2. Topological T-duality and the dilaton shift. The aim of this section is to revisit some aspects of *topological T-duality* for principal torus bundles, as defined in [7] and further studied in [12]. The novelty here is to understand the notion of duality for pairs (V_+, div) in Definition 6.2 in this context, with the upshot of a

mathematical explanation of the *dilaton shift* in string theory, as originally observed by Buscher [9] (see Remark 5.5).

First, let us briefly summarise the basic definition following closely [12]. We are in the situation of the previous section with (E, M, T) , $(\hat{E}, \hat{M}, \hat{T})$ a pair of equivariant Courant algebroids over a common base B . In this section we assume that E and \hat{E} are exact. We denote by

$$[H] \in H^3(M, \mathbb{R})^T, \quad [\hat{H}] \in H^3(\hat{M}, \mathbb{R})^{\hat{T}}$$

the Ševera classes of E and \hat{E} , respectively. Consider the fibre product $\overline{M} = M \times_B \hat{M}$ and the diagram

$$\begin{array}{ccc} & \overline{M} & \\ \bar{p} \swarrow & & \searrow \hat{p} \\ M & & \hat{M} \\ p \searrow & & \swarrow \hat{p} \\ & B & \end{array}$$

Following [7], we say that $(M, [H])$ is T-dual to $(\hat{M}, [\hat{H}])$ if there exists representants H and \hat{H} of the Ševera classes such that

$$\bar{p}^* H - \hat{p}^* \hat{H} = d\overline{B},$$

where $\overline{B} \in \Gamma(\Lambda^2 T^* \overline{M})$ is a $T \times \hat{T}$ -invariant 2-form such that

$$B: \text{Ker } d\bar{p} \otimes \text{Ker } d\hat{p} \rightarrow \mathbb{R}$$

is non-degenerate. Typically T and \hat{T} are taken to be mutually dual tori, but this condition is irrelevant for our discussion. We now recall the salient implications of the T-duality relation [7, 12], in a way that is convenient for the present work.

Theorem 6.6 ([7, 12]). *If $(M, [H])$ and $(\hat{M}, [\hat{H}])$ are T-dual, then there exists an isomorphism of Courant algebroids between the simple reductions*

$$\psi: E/T \rightarrow \hat{E}/\hat{T},$$

and an isomorphism between the spaces of sections of the twisted spinor bundles

$$\tau: \Gamma(\mathbb{S}) \rightarrow \Gamma(\hat{\mathbb{S}}),$$

which exchanges the canonical Dirac generating operators in Theorem 2.7 and which is compatible with Clifford multiplication, that is,

$$\tau(\not{d}_0 \alpha) = \hat{\not{d}}_0(\tau \alpha), \quad \text{and} \quad \tau(e \cdot \alpha) = \psi(e) \cdot \tau(\alpha)$$

for all $e \in \Gamma(E)^T$ and $\alpha \in \Gamma(\mathbb{S})$.

Recall from Example 2.8 that $(\Gamma(\mathbb{S}), \not{d}_0)$ corresponds to the differential complex $(\Gamma(\Lambda^* T^* M), d_H)$, with d_H the twisted de Rham differential. As a direct consequence of the previous theorem, topological T-duals provide a strong version of T-duals in the sense of Definition 6.1. An interesting difference between Theorem 6.6 and Definition 6.1 is that T-duality—given by the isomorphism ψ —is a phenomenon which occurs on the common base B , while topological T-duality exchanges refined information on the total spaces of the fibrations via τ . In fact, understanding the isomorphism τ in an invariant way from the point of view of the base is cumbersome, as the spinor bundle \mathbb{S} involves a twist by the half-density bundle $|\det T^* M|^{\frac{1}{2}}$. This mismatch between the nature of the isomorphisms τ and ψ , is the responsible

of a phenomenon known in the literature as *dilaton shift*, that we try to explain now using Definition 6.2.

To provide a more invariant description, we shall work in the generality of Definition 6.1 and Definition 6.2, without using any explicit formula for the duality isomorphism. Let (V_+, div) and $(\hat{V}_+, \hat{\text{div}})$ be invariant dual pairs on exact equivariant Courant algebroids E and \hat{E} , in the sense of Definition 6.2. We assume that the generalized metric V_+ satisfies $V_+ \cap T^*M = \{0\}$, so that it induces standard T -invariant metric g on M . Since g is an invariant metric on M , it is equivalent to a triple $(\underline{g}, \theta, h)$, where \underline{g} is a metric on B , $\theta: TM \rightarrow VM = \text{Ker } p$ is a connection on $M \rightarrow B$, and h is a metric on the fibres of M , that is,

$$g = h(\theta, \theta) + p^* \underline{g}.$$

Using the explicit calculation in Section 3.3, we can express

$$\text{div}(e') = \mu_g^{-1} L_{\pi_E(e')} \mu_g - \langle e, e' \rangle \quad (6.3)$$

for a suitable $e \in \Gamma(E)^T$, where μ_g denotes the pseudo-Riemannian density. We denote by $\nu \in \Gamma(|\det VM^*|)$ the density induced by h on the vertical bundle $VM \subset TM$. Identifying

$$VM \cong M \times \mathfrak{t},$$

where \mathfrak{t} denotes the Lie algebra of T , we regard $\nu = |\det h|$ as an equivariant map $\nu: M \rightarrow |\det \mathfrak{t}^*|$ and, using that T is unimodular, we obtain a well-defined map

$$\nu: B \rightarrow |\det \mathfrak{t}^*|.$$

Definition 6.7. The *shifting 1-form* of the T -invariant generalized metric V_+ is

$$d \log \nu := \nu^{-1} d\nu \in \Gamma(T^*B).$$

Similarly, assuming that $\hat{V}_+ \cap T^*\hat{M} = \{0\}$, we denote by $(\hat{e}, \hat{g}, \hat{\underline{g}}, \hat{\theta}, \hat{h}, \hat{\nu})$ the corresponding tuple. Note that Definition 6.2 implies that the induced metrics on the base coincide $\hat{\underline{g}} = \underline{g}$.

Proposition 6.8 (Dilaton shift). *With the notation above, the equality of the divergence operators $p_* \text{div} = \psi^* \hat{p}_* \hat{\text{div}}$ in Definition 6.2 is equivalent to*

$$\hat{e} = \psi(e) + 2d \log(\hat{\nu}/\nu), \quad (6.4)$$

where $d \log(\hat{\nu}/\nu)$ is regarded as a section of \hat{E}/\hat{T} using the anchor map.

Proof. We denote $\underline{\pi} = \pi_{E/T}$. The statement follows from the formula

$$\text{div}(e') = \mu_{\underline{g}}^{-1} L_{\underline{\pi}(e')} \mu_{\underline{g}} + 2\langle \sigma_\nu, e' \rangle - \langle e, e' \rangle,$$

for any $e' \in \Gamma(E)^T$, since the metric on B induced by g and \hat{g} coincide and ψ is the identity along the Kernel of the anchor map $\underline{\pi}$. To prove this formula, we calculate

$$\mu_g^{-1} L_{\pi_E(e')} \mu_g = \mu_{\underline{g}}^{-1} L_{\underline{\pi}(e')} \mu_{\underline{g}} + \nu^{-1} d\nu(\underline{\pi}(e')),$$

using the natural decomposition $\mu_g = \mu_{\underline{g}} \otimes \nu$. \square

Note that if $e = \varphi$ and $\hat{e} = \hat{\varphi}$ for invariant 1-forms φ on M and $\hat{\varphi}$ on \hat{M} , respectively, then (6.4) is equivalent to

$$\hat{\varphi} = \varphi - 2d \log \left(\frac{|\det h|}{|\det \hat{h}|} \right). \quad (6.5)$$

Going back to the setup of topological T-duality, using now the explicit *Buscher rules* for the generalized metric we have (see [12])

$$|\det \hat{h}| = |\det h|^{-1}.$$

Thus, equation (6.5) tell us the precise way in which the ‘Weyl part’ of the pure-type operators of (V_+, div) , given in (3.17), changes under the duality isomorphism. Finally, assuming that φ and $\hat{\varphi}$ are exact, with the normalization $\varphi = 8d\phi$ and $\hat{\varphi} = 8d\hat{\phi}$ we obtain

$$\hat{\phi} = \phi - \frac{1}{2} \log |\det h|.$$

This equation is certainly not new: it was encountered by physicists in their computations of the dual Riemannian metric and dilaton field for T-dual sigma-models [9, 10] and carries the name of *dilaton shift* (see Remark 5.9). We should stress that our formula (6.4) is independent of the explicit form of the duality isomorphism, and it is still valid if we replace T and \hat{T} in Definition 6.1 and Definition 6.2 by arbitrary non-abelian unimodular Lie groups. Based on this, we believe that (6.4) generalizes to the non-abelian setup of Poisson-Lie T-duality [36]. It would be interesting to compare our formula (6.4) with [16, Eq. (3.16)], in the context of non-abelian duality in physics.

A different point of view on formula (6.4) is provided by the theory of Dirac generating operators, as considered in Section 2.2. Tracing back the construction of generating operators in the proof of Proposition 2.10 and using Lemma 6.4, one can check that there is an identification between T -invariant generating operators on $\Gamma(\mathcal{S})$ of the Courant algebroid E and generating operators on $\Gamma(\mathcal{S})^T = \Gamma(\mathcal{S}/T)$ of E/T . Using now the decomposition $\mu_g = \mu_{\hat{g}} \otimes \nu$ of the Riemannian density, we obtain a commutative diagram of vector space isomorphisms

$$\begin{array}{ccc} \Gamma(\mathcal{S})^T & \xrightarrow{\otimes \mu_{\hat{g}}^{-\frac{1}{2}}} & \Gamma(\mathbb{S})^T \\ \downarrow p_* & & \downarrow \otimes \nu^{-\frac{1}{2}} \\ \Gamma(\mathcal{S}/T) & \xrightarrow{\otimes \mu_{\hat{g}}^{-\frac{1}{2}}} & \Gamma(\mathbb{S}_{E/T}) \end{array} \quad (6.6)$$

Using the horizontal arrows, we pull-back the canonical generating operator \not{d}_0 on $\Gamma(\mathbb{S})^T$ and \not{d}_0 on $\Gamma(\mathbb{S}_{E/T})$ to the space of T -invariant sections $\Gamma(\mathcal{S})^T$ and to $\Gamma(\Gamma(\mathcal{S}/T))$, respectively. Then, one has

$$p_*(\not{d}_0) = \not{d}_0 + \frac{1}{2} d \log \nu \cdot.$$

Consequently, (6.4) can be interpreted as the discrepancy between $p_*(\not{d}_0)$ and the push-forward of the dual $p_*(\hat{\not{d}}_0)$ using the isomorphism $\Gamma(\mathbb{S}_{E/T}) \cong \Gamma(\hat{\mathbb{S}}_{\hat{E}/\hat{T}})$ induced by τ .

7. THE TRANSITIVE CASE

In this section we apply our framework to the case of transitive Courant algebroids obtained by reduction. In particular, building on a result in [23], we prove that the Strominger system of partial differential equations is invariant under T-duality (see Definition 6.1).

Let G be a Lie group, endowed with a biinvariant non-degenerate pairing on its Lie algebra, which we denote

$$c: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathbb{R}.$$

Let $p: P \rightarrow M$ be a principal G -bundle over a smooth manifold M of dimension n . Provided that the (real) first Pontryagin class of P with respect to c vanishes

$$p_1^c(P) = 0 \in H^4(M, \mathbb{R}),$$

there exists a transitive Courant algebroid E over $M = P/G$ given by an extension

$$0 \rightarrow T^*M \rightarrow E \rightarrow TP/G \rightarrow 0. \quad (7.1)$$

The anchor map of E is given by the natural projection $TP/G \rightarrow TM$ and there is an isomorphism

$$(\text{ad } P, c) \cong \text{Ker } \pi_E / (\text{Ker } \pi_E)^\perp$$

as bundles of quadratic Lie algebras. The isomorphism classes of transitive Courant algebroids of this form are parametrized by the $H^3(M, \mathbb{R})$ -torsor of *real string classes* on P [5]

$$H_{str}^3(P, \mathbb{R}) \subset H^3(P, \mathbb{R}).$$

Recall that a cohomology class $[\hat{H}] \in H^3(P, \mathbb{R})$ is said to be a string class if the restriction of $[\hat{H}]$ to the fibres of P coincides with the Cartan 3-form class in $H^3(G, \mathbb{R})$ determined by the pairing c .

We fix a string class $[\hat{H}] \in H_{str}^3(P, \mathbb{R})$, and consider the associated transitive Courant algebroid E . Given a connection $\theta \in P$, there exists an isotropic splitting of E and an isomorphism (see [21, Proposition 2.4])

$$E = T \oplus \text{ad } P \oplus T^*,$$

such that the pairing is given by

$$\langle X + r + \xi, Y + t + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y)) + c(r, t),$$

the anchor is the canonical projection $\pi_E(X + r + \xi) = X$ and the bracket is [42]

$$\begin{aligned} [X + r + \xi, Y + t + \eta] &= [X, Y] + L_X \eta - \iota_Y d\xi + \iota_Y i_X H \\ &\quad - [r, t] - F(X, Y) + d_X^\theta t - d_Y^\theta r \\ &\quad + 2c(d_Y^\theta r, t) + 2c(\iota_X F, t) - 2c(\iota_Y F, r). \end{aligned} \quad (7.2)$$

In the last expression, H is a 3-form on M (determined up to addition of exact 2-forms) and d^θ is the covariant derivative on $\text{ad } P$ induced by θ . The condition that (7.2) satisfies the Jacobi identity is equivalent to the *Bianchi identity*

$$dH = c(F \wedge F), \quad (7.3)$$

which gives a distinguished trivialization of the first Pontryagin class of the principal bundle $p_1^c(P) = [c(F \wedge F)] \in H^4(M, \mathbb{R})$. We note that the pair (H, θ) determines a representant

$$\hat{H} = p^*H - CS(\theta) \in H_{str}^3(P, \mathbb{R}), \quad (7.4)$$

where $CS(\theta) = -\frac{1}{6}c(\theta \wedge [\theta, \theta]) + c(F \wedge \theta)$ is the Chern-Simons three-form of the connection θ .

A generalized metric $V_+ \subset E$ is said to be *admissible* if $r_+ = n$ and V_- intersects T^* transversally, that is, $V_- \cap T^* = \{0\}$ (notice the different sign notation in [21, Definition 3.1]). An admissible generalized metric determines an isotropic splitting of E , a connection θ on P and a three-form H on M , so that the Courant structure is as above and we have the following simple expression for V_\pm

$$\begin{aligned} V_+ &= \{X + g(X) : X \in \Gamma(T)\}, \\ V_- &= \{X - g(X) + r : X \in \Gamma(T), r \in \Gamma(\text{ad } P)\}, \end{aligned}$$

where g is a (standard) metric on M . In the sequel we will assume that g is Riemannian.

A torsion-free generalized connection for the admissible metric V_+ was constructed in [23] (see also [21]), and it is given explicitly by

$$\begin{aligned} D_{a_-}^0 c_- &= 2\Pi_- \left(\nabla_X^{-1/3} Z - \frac{2}{3} g^{-1} c(i_X F, t) - \frac{1}{3} g^{-1} c(i_Z F, r) \right) \\ &\quad + d_X^\theta t - \frac{2}{3} F(X, Z) - \frac{1}{3} c^{-1} c(r, [t, \cdot]), \\ D_{b_+}^0 c_- &= 2\Pi_- (\nabla_Y^- Z - g^{-1} c(i_Y F, t)) + d_Y^\theta t - F(Y, Z), \\ D_{a_-}^0 b_+ &= 2\Pi_+ (\nabla_X^+ Y - g^{-1} c(i_Y F, r)), \\ D_{b_+}^0 d_+ &= 2\Pi_+ (\nabla_Y^{1/3} W). \end{aligned} \tag{7.5}$$

where

$$\begin{aligned} a_- &= X + r - gX, \\ b_+ &= Y + gY, \\ c_- &= Z + t - gZ, \\ d_+ &= W + gW. \end{aligned} \tag{7.6}$$

By [23, Eq. (5.16)] it follows that D^0 differs from the generalized connection D induced by

$$\nabla^g \oplus d^\theta \oplus \nabla^{g*}$$

by a totally skew-symmetric element, and therefore $\text{div}_{D^0} = \text{div}_D$. Now a direct calculation using (3.14) shows that

$$\text{div}_{D^0}(e) = \text{div}_D(e) = \mu_g^{-1} L_{\pi(e)} \mu_g.$$

Denote $\text{div} = \text{div}_{D^0}$ and consider

$$\text{div}^\varphi = \text{div} - \langle \varphi, \cdot \rangle, \tag{7.7}$$

for a 1-form $\varphi \in \Gamma(T^*)$. Then, we can construct an element $D^\varphi \in \mathcal{D}(V_+, \text{div}^\varphi)$ by

$$D^\varphi = D^0 + \frac{1}{n-1} \chi_+^{\varphi^+} + \frac{1}{r_- - 1} \chi_-^{\varphi^-} \tag{7.8}$$

The following formulae for the Ricci tensors follows from a lengthy calculation using the curvatures of the generalized connection D^φ .

Lemma 7.1.

$$\begin{aligned} \text{Ric}^+(c^-, b^+) &= \left(\text{Ric}^{\nabla^+} - F \circ F - \frac{1}{4} \nabla^+ \varphi \right) (Y, Z) \\ &\quad - i_Y c \left(d_A^* F - \frac{1}{2} * (F \wedge * H) + \frac{1}{4} F(g^{-1} \varphi, \cdot), t \right), \\ \text{Ric}^-(b^+, c^-) &= \left(\text{Ric}^{\nabla^-} - F \circ F - \frac{1}{4} \nabla^- \varphi \right) (Y, Z) \\ &\quad + i_Y c \left(d_A^* F - \frac{1}{2} * (F \wedge * H) + \frac{1}{4} F(g^{-1} \varphi, \cdot), t \right). \end{aligned}$$

By inspection of the previous formula we note the following.

Proposition 7.2. *Let E be a transitive Courant algebroid determined by $\hat{H} \in H_{str}^3(P, \mathbb{R})$. Let (V_+, div) denote an admissible generalized metric V_+ and a divergence operator div such that $\text{div} - \text{div}_{D^0} \in \Gamma(T^*)$. Then, (V_+, div) satisfies the skew-symmetry property in Definition 5.2.*

Thus, by Proposition 5.3 the Ricci flow in Definition 5.1 and the generalized Ricci flow (5.2) are equivalent. More explicitly, using the explicit formula for Ric^+ above, the Ricci flow is given by

$$\begin{aligned}\partial_t g - \partial_t b &= -2 \left(\text{Ric}^g - \frac{1}{4} H \circ H - \frac{1}{2} d^* H - F \circ F - \frac{1}{4} \nabla^+ \varphi \right), \\ \partial_t A &= 2 \left(d_A^* F - \frac{1}{2} * (F \wedge * H) + \frac{1}{4} F(g^{-1} \varphi, \cdot) \right).\end{aligned}$$

We next state a result from [23], which establishes the link between the Killing spinor equations (5.6) and the Strominger system [33, 45]. Assume that M is a spin compact manifold of dimension 6. Then, a solution of the Strominger system on (M, P) is given by a tuple (ω, Ω, θ) , given by a $SU(3)$ -structure on M and a connection θ on P , satisfying equations

$$\begin{aligned}d\Omega &= 0, \\ F_\theta \wedge \omega^2 &= 0, \\ d^* \omega - d^c \log \|\Omega\|_\omega &= 0, \\ dd^c \omega - c(F_\theta \wedge F_\theta) &= 0.\end{aligned}\tag{7.9}$$

We refer to the reviews [1, 22] for a detailed discussion about these interesting equations. Note from the last equation that any solution of the Strominger system determines in particular a string class in $[\hat{H}] \in H_{str}^3(P, \mathbb{R})$, given by (7.4) with $H = d^c \omega$. We shall denote $E_{[\hat{H}]}$ the corresponding Courant algebroid.

Theorem 7.3 ([23]). *Assume that M is a spin compact manifold of dimension 6. Solutions of the Strominger with string class $[\hat{H}] \in H_{str}^3(P, \mathbb{R})$ are in correspondence with solutions (V_+, div) of the Killing spinor equations (5.6) on $E_{[\hat{H}]}$, such that V_+ is admissible and $\varphi = \text{div} - \text{div}_{D^0}$ is an exact 1-form.*

The particular instance of the equations (7.9) which is relevant for physics is when the principal bundle P is a fibred product

$$P = P_M \times_M P_K,$$

where P_M is the bundle of oriented frames of M and P_K is principal bundle with compact structure group K , and the pairing c is

$$c = \text{tr}_{\mathfrak{gl}} - \text{tr}_{\mathfrak{k}}$$

for $-\text{tr}_{\mathfrak{gl}}$ and $-\text{tr}_{\mathfrak{k}}$ positive definite forms on $\mathfrak{gl}(6, \mathbb{R})$ and \mathfrak{k} , respectively. In this particular situation, we obtain a strong form of Proposition 5.7, which recovers the ‘if part’ of the main theorem in [34].

Proposition 7.4. *Assume that $P = P_K \times_M P_{K'}$ and that $c = \text{tr}_{\mathfrak{k}} - \text{tr}_{\mathfrak{k}'}$, with $-\text{tr}_{\mathfrak{k}}$ and $-\text{tr}_{\mathfrak{k}'}$ positive definite forms. Then, any solution (V_+, div) of the Killing spinor equations (5.6) with V_+ admissible is Ricci flat.*

Proof. By assumption, an element $s \in \text{ad } P$ decomposes as $s = r + t \in \text{ad } P_K \oplus \text{ad } P_{K'}$. The result follows applying Proposition 5.7 and using the explicit formula for the Ricci tensor in Lemma 7.1. \square

We address now the main result of this section, which follows as a direct consequence of Theorem 7.3, Theorem 6.5 and our formula for the *dilaton shift* in Proposition 6.8.

Theorem 7.5 ([23]). *The solutions of the Strominger system are preserved by T-duality, as in Definition 6.1.*

Proof. By Theorem 7.3 any solution of the Strominger system with string class $[\hat{H}]$ determines a solution of the Killing spinor equations $(V_+, \text{div}^\varphi)$ on a Courant algebroid $E_{[\hat{H}]}$, and with φ exact. If $\hat{E} \rightarrow \hat{M}$ is a dual Courant algebroid (for suitable symmetry groups), then by Theorem 6.5 it also carries a solution of the Killing spinor equations $(\hat{V}_+, \text{div}^{\hat{\varphi}})$. Furthermore, by Proposition 6.8, the 1-forms $\hat{\varphi}$ and φ are related by the *dilaton shift* equation (6.5). Therefore, since φ is exact it follows that $\hat{\varphi}$ is also exact, and by Theorem 7.3 it determines a solution of the Strominger system on the dual space. Note that (6.4) is valid for admissible metrics in the situation considered here. \square

An explicit notion of T-duality for transitive Courant algebroids obtained by reduction and with integral string class has been proposed in [5], under the name of *heterotic T-duality*. By [5, Proposition 4.13] it follows that our Theorem 7.5 applies in this situation. It would be interesting to find examples of heterotic T-duals which solve the Strominger system among the many concrete examples with abelian symmetries that exist in the literature (see references therein [22]). We hope to come back to this question in future work.

REFERENCES

- [1] I. Agricola, *The Srní lectures on non-integrable geometries with torsion*, Arch. Math. (Brno) **42** (2006), 5–84. With an appendix by Mario Kassuba.
- [2] I. Agricola and Th. Friedrich, *On the holonomy of connections with skew-symmetric torsion*, Math. Ann. **328** (2004), 711–748.
- [3] I. Agricola and A. C. Ferreira, *Einstein manifolds with skew torsion*, Q. J. Math. **65** (2014) 717–741.
- [4] A. Alekseev and P. Xu, *Derived brackets and courant algebroids*, unpublished, available at <http://www.math.psu.edu/ping/anton-final.pdf> (2001).
- [5] D. Baraglia and P. Hekmati, *Transitive Courant Algebroids, String Structures and T-duality*, Advances in Theoretical and Mathematical Physics **19** (2015), no. 3, 613–672.
- [6] J.-M. Bismut, *A local index theorem for non Kähler manifolds*, Math. Ann. **289** (1869) 681–699.
- [7] P. Bouwknegt, J. Evslin, and V. Mathai, *T-duality: topology change from H-flux*, Comm. Math. Phys. **249** (2004) 383–415.
- [8] H. Bursztyn, G. Cavalcanti and M. Gualtieri, *Reduction of Courant algebroids and generalized complex structures*, Adv. Math. **211** (2) (2007) 726–765.
- [9] T. Buscher, *A symmetry of the strong background field equations*, Phys. Lett. **B194** (1987) 59–62.
- [10] ———, *Path-integral derivation of quantum duality in nonlinear sigma-models*, Phys. Lett. **B201** (1988) 446–472.
- [11] C. Callan, D. Friedan, E. Martinec and M. Perry, *Strings in background fields*, Nucl. Phys. B **262** (1985) 593–609.
- [12] G. Cavalcanti and M. Gualtieri, *Generalized complex geometry and T-duality*, In *A Celebration of the Mathematical Legacy of Raoul Bott*, Am. Math. Soc. (CRM Proceedings & Lecture Notes) (2010) 341–366.
- [13] A. Clarke, M. Garcia-Fernandez, and C. Tipler *Moduli of G_2 structures and the Strominger system in dimension 7*, arXiv:1607.01219.
- [14] A. Coimbra, R. Minasian, H. Triendl and D. Waldram, *Generalised geometry for string corrections*, JHEP **11** (2014) 160.
- [15] A. Coimbra, C. Strickland-Constable and D. Waldram, *Supergravity as Generalised Geometry I: Type II Theories*, JHEP **11** (2011) 91.

- [16] X. de la Ossa and F. Quevedo, *Duality Symmetries from Non-Abelian Isometries in String Theories*, Nucl. Phys. **B403** (1993) 377–394.
- [17] P. Deligne et al., *Quantum fields and strings: a course for mathematicians II*, American Mathematical Society, 1999.
- [18] M. Fernández, S. Ivanov, L. Ugarte, R. Villacampa, *Non-Kähler heterotic-string compactifications with non-zero fluxes and constant dilaton*, Commun. Math. Phys. **288** (2009) 677–697, 0804.1648 [hep-th].
- [19] A. C. Ferreira, *A vanishing theorem in twisted De Rham cohomology*, arXiv:1012.2087 [math.DG] (2011).
- [20] Th. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian Journ. Math. **6** (2002), 303–336.
- [21] M. Garcia-Fernandez, *Torsion-free generalized connections and heterotic supergravity*, Commun. Math. Phys. **332** (2014) 89–115.
- [22] ———, *Lectures on the Strominger system*, Travaux Mathématiques, Special Issue: School GEOQUANT at the ICMAT, Vol. XXIV (2016) 7–61.
- [23] M. Garcia-Fernandez, R. Rubio and C. Tipler, *Infinitesimal moduli for the Strominger system and Killing spinors in generalized geometry*, Math. Ann. (2016). doi:10.1007/s00208-016-1463-5, arXiv:1503.07562.
- [24] M. Garcia-Fernandez and C. Shahbazi, *Twisted heterotic compactifications and moduli stabilization*, ongoing.
- [25] M. Graña, R. Minasian, M. Petrini, and A. Tomasiello, *Supersymmetric backgrounds from generalized Calabi-Yau manifolds*, JHEP **08** (2004) 046.
- [26] M. Graña, R. Minasian, M. Petrini, and D. Waldram, *T-duality, generalized geometry and non-geometric backgrounds*, JHEP **09** (2009) 075.
- [27] M. Grützmann, J. P. Michel and P. Xu, *Weyl quantization of degree 2 symplectic graded manifolds*, arXiv:1410.3346 [math.DG].
- [28] M. Gualtieri, *Generalized Complex Geometry*, DPhil thesis, University of Oxford (2004), arXiv: 0401221.
- [29] ———, *Generalized Complex Geometry*, Ann. Math. **174** (2011) 75–123.
- [30] ———, *Branes on Poisson varieties, The many facets of geometry*, 368–394, Oxford Univ. Press, Oxford (2010).
- [31] N. Hitchin, *Generalized Calabi-Yau manifolds*, Q. J. Math. **54** (2003) 281–308.
- [32] ———, *The moduli space of special Lagrangian submanifolds*, Dedicated to Ennio DeGiorgi. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **25** (1997) 503–515.
- [33] C. Hull, *Superstring compactifications with torsion and space-time supersymmetry*, In Turin 1985 Proceedings “Superunification and Extra Dimensions” (1986) 347–375.
- [34] S. Ivanov, *Heterotic supersymmetry, anomaly cancellation and equations of motion*, Phys. Lett. B **685** (2-3) (2010) 190–196, 0908.2927 [hep-th].
- [35] S. Ivanov and G. Papadopoulos, *Vanishing theorems and string backgrounds*, Class. Quant. Grav. **18** (2001) 1089–1110.
- [36] C. Klimčík and P. Severa, *Dual Non-Abelian Duality and the Drinfeld Double*, Phys. Lett. **B351** (1995) 455–462.
- [37] J. Li and S.-T. Yau, *The existence of supersymmetric string theory with torsion*, J. Diff. Geom. **70** (2005) 143–181.
- [38] R. Loja Fernandes, *Invariants of Lie algebroids*, Diff. Geom. Appl. **19** (2003) 223–243.
- [39] M. Roček and E. Verlinde, *Duality, quotients, and currents*, Nuclear Phys. **B373** (1992) 630–646.
- [40] R. Rubio, private communication (2016).
- [41] S. Salamon, *A Tour of Exceptional Geometry*, Milan j. math. **71** (2003) 59–94.
- [42] P. Ševera, *Fifth letter to Alan Weinstein*, <http://sophia.dtp.fmph.uniba.sk/severa/>.
- [43] P. Ševera and F. Valach, *Ricci flow, Courant algebroids, and renormalization of Poisson–Lie T-duality*, arXiv:1610.09004 [math.DG] (2016).
- [44] J. Streets, *Generalized geometry, T-duality, and renormalization group flow*, arXiv:1310.5121 (2013).
- [45] A. Strominger, *Superstrings with torsion*, Nucl. Phys. B **274** (2) (1986) 253–284.
- [46] A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror symmetry is T-Duality*, Nuclear Phys. **B479** (1996) 243–259.
- [47] F. Witt, *Generalised G2-manifolds*, Commun. Math. Phys. **265** (2006) 275–303.

- [48] S.-T. Yau, *Complex geometry: Its brief history and its future*, Science in China Series A Mathematics **48** (2005) 47–60.

INSTITUTO DE CIENCIAS MATEMÁTICAS (CSIC-UAM-UC3M-UCM), NICOLÁS CABRERA 13–15, CANTOBLANCO, 28049 MADRID, SPAIN
E-mail address: `mario.garcia@icmat.es`